www.artofproblemsolving.com/community/c583596

by laegolas, DK, Rushil, stvs_f, momo1729, darij grinberg

– Paper 1

1 Are there integers x, y and z which satisfy the equation

$$z^2 = (x+1)(y^2-1)+n$$

when (a) $n = 2006$ (b) $n = 2007$?

2 P and Q are points on the equal sides AB and AC respectively of an isosceles triangle ABC such that $AP = CQ$. Moreover, neither P nor Q is a vertex of ABC. Prove that the circumcircle of the triangle APQ passes through the circumcenter of the triangle ABC.

3 Prove that a square of side 2.1 units can be completely covered by seven squares of side 1 unit.

Extra: Try to prove that 7 is the minimal amount.

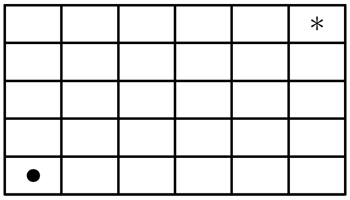
4 Find the greatest value and the least value of $x+y$ where x, y are real numbers, with $x \geq -2$, $y \geq -3$ and

$$x - 2\sqrt{x+2} = 2\sqrt{y+3} - y$$

5 Find all functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that $f(xy + f(x)) = xf(y) + f(x)$ for all $x, y \in \mathbb{R}$.

– Paper 2

1 The rooms of a building are arranged in a $m \times n$ rectangular grid (as shown below for the 5×6 case). Every room is connected by an open door to each adjacent room, but the only access to or from the building is by a door in the top right room. This door is locked with an elaborate system of mn keys, one of which is located in every room of the building. A person is in the bottom left room and can move from there to any adjacent room. However, as soon as the person leaves a room, all the doors of that room are instantly and automatically locked. Find, with proof, all m and n for which it is possible for the person to collect all the keys and escape the building.



• starting position

∗ room with locked external door

---

2 ABC is a triangle with points D, E on BC with D nearer B; F, G on AC, with F nearer C; H, K on AB, with H nearer A. Suppose that $AH = AG = 1$, $BK = BD = 2$, $CE = CF = 4$, $\angle B = 60^\circ$ and that D, E, F, G, H and K all lie on a circle. Find the radius of the incircle of triangle ABC.

---

3 let x,y are positive and $\in R$ that : $x+2y=1$.prove that :

$$\frac{1}{x}+\frac{2}{y}\geq\frac{25}{1+48xy^2}$$

---

4 Let n be a positive integer.
Find the greatest common divisor of the numbers $\binom{2n}{1},\binom{2n}{3},\binom{2n}{5},...,\binom{2n}{2n-1}$.

---

5 Let n and k be positive integers. There are given n circles in the plane. Every two of them intersect at two distinct points, and all points of intersection they determine are pairwise distinct (i. e. no three circles have a common point). No three circles have a point in common. Each intersection point must be colored with one of n distinct colors so that each color is used at least once and exactly k distinct colors occur on each circle. Find all values of $n \geq 2$ and k for which such a coloring is possible.

Proposed by Horst Sewerin, Germany
