Mathematical Olympiad 2017

www.artofproblemsolving.com/community/c584444

by ManuelKahayon

1 Given $n \in \mathbb{N}$, let $\sigma(n)$ denote the sum of the divisors of n and $\phi(n)$ denote the number of integers $n \geq m$ for which $\gcd(m,n) = 1$. Show that for all $n \in \mathbb{N}$,

$$\frac{1}{\sigma(n)} + \frac{1}{\phi(n)} \geq \frac{2}{n}$$

and determine when equality holds.

---

2 Find all positive real numbers $(a,b,c) \leq 1$ which satisfy

$$\min\left\{\sqrt{\frac{ab+1}{abc}}\sqrt{\frac{bc+1}{abc}}, \sqrt{\frac{ac+1}{abc}}\right\} = \sqrt{\frac{1-a}{a}} + \sqrt{\frac{1-b}{b}} + \sqrt{\frac{1-c}{c}}$$

---

3 Each of the numbers in the set $A = \{1, 2, \cdots, 2017\}$ is colored either red or white. Prove that for $n \geq 18$, there exists a coloring of the numbers in A such that any of its n-term arithmetic sequences contains both colors.

---

4 Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ with centers at C_1 and C_2 respectively, intersect at two points A and B. Points P and Q are varying points on $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, such that P, Q and B are collinear and B is always between P and Q. Let lines PC_1 and QC_2 intersect at R, let I be the incenter of ΔPQR, and let S be the circumcenter of ΔPIQ. Show that as P and Q vary, S traces the arc of a circle whose center is concyclic with A, C_1 and C_2.
