JBMO Shortlist 2014

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– Algebra

1 Solve in positive real numbers: $n+\lfloor \sqrt{n} \rfloor+\lfloor \sqrt[3]{n} \rfloor=2014$

2 Let a,b,c be positive real numbers such that $abc=\frac{1}{8}$. Prove the inequality:

$$a^2+b^2+c^2+a^2b^2+b^2c^2+c^2a^2\geq \frac{15}{16}$$

When the equality holds?

3 For positive real numbers a,b,c with $abc=1$ prove that $\left(a+\frac{1}{b}\right)^2+\left(b+\frac{1}{c}\right)^2+\left(c+\frac{1}{a}\right)^2\geq 3(a+b+c+1)$

4 With the conditions $a,b,c\in \mathbb{R}^+$ and $a+b+c=1$, prove that

$$\frac{7+2b}{1+a}+\frac{7+2c}{1+b}+\frac{7+2a}{1+c}\geq \frac{69}{4}$$

5 Let x,y and z be non-negative real numbers satisfying the equation $x+y+z=xyz$. Prove that $2(x^2+y^2+z^2)\geq 3(x+y+z)$.

6 Let a,b,c be positive real numbers. Prove that

$$\left((3a^2+1)^2+2\left(1+\frac{3}{b}\right)^2\right)\left((3b^2+1)^2+2\left(1+\frac{3}{c}\right)^2\right)\left((3c^2+1)^2+2\left(1+\frac{3}{a}\right)^2\right)\geq 48^3$$

7 $a,b,c\in \mathbb{R}^+$ and $a^2+b^2+c^2=48$. Prove that

$$a^2\sqrt{2b^3+16}+b^2\sqrt{2c^3+16}+c^2\sqrt{2a^3+16}\leq 24^2$$

8 Let x,y,z be positive real numbers such that $xyz=1$. Prove the inequality:

$$\frac{1}{x\left(ay+b\right)}+\frac{1}{y\left(az+b\right)}+\frac{1}{z\left(ax+b\right)}\geq 3$$

if:
(A) $a=0, b=1$
(B) $a=1, b=0$
(C) $a+b=1,\ a, b>0$
When the equality holds?

9 Let n a positive integer and let $x_1, \ldots, x_n, y_1, \ldots, y_n$ real positive numbers such that $x_1+\ldots+x_n=y_1+\ldots+y_n=1$. Prove that:

$$|x_1-y_1|+\ldots+|x_n-y_n| \leq 2-\underset{1\leq i\leq n}{min}\frac{x_i}{y_i}-\underset{1\leq i\leq n}{min}\frac{y_i}{x_i}$$

– Combinatorics

1 There are some real numbers on the board (at least two). In every step we choose two of them, for example a and b, and then we replace them with $\frac{ab}{a+b}$. We continue until there is one number. Prove that the last number does not depend on which order we choose the numbers to erase.

2 In a country with n towns, all the direct flights are of double destinations (back and forth). There are $r>2014$ rootes between different pairs of towns, that include no more than one intermediate stop (direction of each root matters). Find the minimum possible value of n and the minimum possible r for that value of n.

3 For a positive integer n, two payers A and B play the following game: Given a pile of s stones, the players take turn alternatively with A going first. On each turn the player is allowed to take either one stone, or a prime number of stones, or a positive multiple of n stones. The winner is the one who takes the last stone. Assuming both A and B play perfectly, for how many values of s the player A cannot win?

4 $A=1\cdot 4\cdot 7\cdots 2014$.Find the last non-zero digit of A if it is known that $A\equiv 1 \mod 3$.

– Geometry

1 Let ABC be a triangle with $m\left(\angle B\right)=m\left(\angle C\right)=40^{\circ}$ Line bisector of $\angle B$ intersects AC at point D. Prove that $BD+DA=BC$.

2 Acute-angled triangle ABC with $AB<AC<BC$ and let be $c(O,R)$ it's circumcircle. Diameters BD and CE are drawn. Circle $c_1(A,AE)$ interescts AC at K. Circle $c_2(A,AD)$ intersects BA at L .(A lies between B and L). Prove that lines EK and DL intersect at circle c .

by Evangelos Psychas (Greece)

3 Consider an acute triangle ABC of area S. Let $CD \perp AB$ ($D \in AB$), $DM \perp AC$ ($M \in AC$) and $DN \perp BC$ ($N \in BC$). Denote by H_1 and H_2 the orthocentres of the triangles MNC, respectively MND. Find the area of the quadrilateral AH_1BH_2 in terms of S.

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4 Let ABC be an acute triangle such that $AB \neq AC$.Let M be the midpoint BC, H the orthocenter of $\triangle ABC, O_1$ the midpoint of AH and O_2 the circumcenter of $\triangle BCH$. Prove that O_1AMO_2 is a parallelogram.

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5 Let ABC be a triangle with $AB \neq BC$; and let BD be the internal bisector of $\angle ABC,$, $(D \in AC)$. Denote by M the midpoint of the arc AC which contains point B. The circumscribed circle of the triangle $\triangle\ BDM$ intersects the segment AB at point $K \neq B$. Let J be the reflection of A with respect to K. If $DJ \cap AM = \{O\}$, prove that the points J, B, M, O belong to the same circle.

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6 Let $ABCD$ be a quadrilateral whose diagonals are not perpendicular and whose sides AB and CD are not parallel.Let O be the intersection of its diagonals.Denote with H_1 and H_2 the orthocenters of triangles AOB and COD, respectively.If M and N are the midpoints of the segment lines AB and $CD,$ respectively.Prove that the lines H_1H_2 and MN are parallel if and only if $AC = BD.$

– Number Theory

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1 All letters in the word $VUQAR$ are different and chosen from the set $\{1, 2, 3, 4, 5\}$. Find all solutions to the equation

$$\frac{(V+U+Q+A+R)^2}{V-U-Q+A+R} = V^{{{U^Q}^A}^R}.$$

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2 Find all triples of primes (p, q, r) satisfying $3p^4 - 5q^4 - 4r^2 = 26$.

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3 Find all integer solutions to the equation $x^2 = y^2(x + y^4 + 2y^2)$.

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4 Prove that there are not intgers a and b with conditions,
i) $16a - 9b$ is a prime number.
ii) ab is a perfect square.
iii) $a + b$ is also perfect square.

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5 Find all non-negative solutions to the equation $2013^x + 2014^y = 2015^z$

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6 Vukasin, Dimitrije, Dusan, Stefan and Filip asked their teacher to guess three consecutive positive integers, after these true statements:
Vukasin: " The sum of the digits of one number is prime number. The sum of the digits of another of the other two is, an even perfect number.(n is perfect if $\sigma(n) = 2n$). The sum of the digits of the third number equals to the number of it's positive divisors".

Dimitrije:"Everyone of those three numbers has at most two digits equal to 1 in their decimal representation".
Dusan:"If we add 11 to exactly one of them, then we have a perfect square of an integer"
Stefan:"Everyone of them has exactly one prime divisor less than 10".
Filip:"The three numbers are square free".
Professor found the right answer. Which numbers did he mention?