JBMO Shortlist 2009

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–	Algebra
1	Determine all integers a, b, c satisfying identities: $a+b+c=15$ $(a-3)^3+(b-5)^3+(c-7)^3=540$
2	$\boxed{A2}$ Find the maximum value of $z+x$ if x, y, z are satisfying the given conditions.$x^2+y^2=4$ $z^2+t^2=9$ $xt+yz\geq 6$
3	Find all values of the real parameter a, for which the system $(\|x\|+\|y\|-2)^2=1$ $y=ax+5$ has exactly three solutions
4	Let x, y, z be real numbers such that $0<x,y,z<1$ and $xyz=(1-x)(1-y)(1-z)$. Show that at least one of the numbers $(1-x)y,(1-y)z,(1-z)x$ is greater than or equal to $\frac{1}{4}$
5	$\boxed{\text{A5}}$ Let x, y, z be positive reals. Prove that $(x^2+y+1)(x^2+z+1)(y^2+x+1)(y^2+z+1)(z^2+x+1)(z^2+y+1)\geq(x+y+z)^6$
–	Combinatorics
1	Each one of 2009 distinct points in the plane is coloured in blue or red, so that on every blue-centered unit circle there are exactly two red points. Find the gratest possible number of blue points.
2	Five players (A,B,C,D,E) take part in a bridge tournament. Every two players must play (as partners) against every other two players. Any two given players can be partners not more than once per a day. What is the least number of days needed for this tournament?
3	a) In how many ways can we read the word SARAJEVO from the table below, if it is allowed to jump from cell to an adjacent cell (by vertex or a side) cell? b) After the letter in one cell was deleted, only 525 ways to read the word SARAJEVO remained. Find all possible positions of that cell.
4	Determine all pairs of (m,n) such that is possible to tile the table $m\times n$ with figure "corner" as in figure with condition that in that tilling does not exist rectangle (except $m\times n$) regularly covered with figures.
–	Geometry

1	Parallelogram $ABCD$ is given with $AC > BD$, and O intersection point of AC and BD. Circle with center at $O$and radius OA intersects extensions of $AD$and ABat points G and L, respectively. Let Z be intersection point of lines $BD$and GL. Prove that $\angle ZCA = 90^{\circ}$.
2	In right trapezoid $ABCD\,(AB \parallel CD)$ the angle at vertex B measures 75°. Point $H$is the foot of the perpendicular from point A to the line BC. If $BH = DC$ and$AD + AH = 8$, find the area of $ABCD$.
3	Parallelogram $ABCD$with obtuse angle $\angle ABC$ is given. After rotation of the triangle ACD around the vertex C, we get a triangle $CD'A'$, such that points B, C and $D'$are collinear. Extensions of median of triangle $CD'A'$ that passes through D'intersects the straight line BD at point P. Prove that $PC$is the bisector of the angle $\angle BPD'$.
4	Let $ABCDE$ be a convex pentagon such that $AB + CD = BC + DE$ and k a circle with center on side AE that touches the sides AB, BC, CD and DE at points P, Q, R and S (different from vertices of the pentagon) respectively. Prove that lines PS and AE are parallel.
5	Let A, B, C and O be four points in plane, such that $\angle ABC > 90^{\circ}$ and $OA = OB = OC$.Define the point $D \in AB$ and the line l such that $D \in l, AC \perp DC$ and $l \perp AO$. Line l cuts ACat E and circumcircle of ABC at F. Prove that the circumcircles of triangles $BEF$and $CFD$are tangent at F.
–	Number Theory
1	Solve in non-negative integers the equation $2^a 3^b + 9 = c^2$
2	A group of $n > 1$ pirates of different age owned total of 2009 coins. Initially each pirate (except the youngest one) had one coin more than the next younger. a) Find all possible values of n. b) Every day a pirate was chosen. The chosen pirate gave a coin to each of the other pirates. If $n = 7$, find the largest possible number of coins a pirate can have after several days.
3	Find all pairs (x, y) of integers which satisfy the equation $(x + y)^2(x^2 + y^2) = 2009^2$
4	Determine all prime numbers $p_1, p_2, ..., p_{12}, p_{13}, p_1 \leq p_2 \leq ... \leq p_{12} \leq p_{13}$, such that $p_1^2 + p_2^2 + ... + p_{12}^2 = p_{13}^2$ and one of them is equal to $2p_1 + p_9$.
5	Show that there are infinitely many positive integers c, such that the following equations both have solutions in positive integers: $(x^2 - c)(y^2 - c) = z^2 - c$ and $(x^2 + c)(y^2 - c) = z^2 - c$.