JBMO Shortlist 2008

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– Algebra

1 If for the real numbers x, y, z, k the following conditions are valid, $x \neq y \neq z \neq x$ and $x^3 + y^3 + k(x^2 + y^2) = y^3 + z^3 + k(y^2 + z^2) = z^3 + x^3 + k(z^2 + x^2) = 2008$, fi nd the product xyz.

2 Find all real numbers a, b, c, d such that

$$\left\{\begin{array}{c} a+b+c+d=20, \\ ab+ac+ad+bc+bd+cd=150. \end{array}\right.$$

3 Let the real parameter p be such that the system $\begin{cases} p(x^2 - y^2) = (p^2 - 1)xy \\ |x-1| + |y| = 1 \end{cases}$ has at least three different real solutions. Find p and solve the system for that p.

4 Find all triples (x, y, z) of real numbers that satisfy the system $\begin{cases} x+y+z=2008 \\ x^2+y^2+z^2=6024^2 \\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2008} \end{cases}$

5 Find all triples (x, y, z) of real positive numbers, which satisfy the system $\begin{cases} \frac{1}{x}+\frac{4}{y}+\frac{9}{z}=3 \\ x+y+z \leq 12 \end{cases}$

6 If the real numbers a, b, c, d are such that $0 < a, b, c, d < 1$, show that $1+ab+bc+cd+da+ac+bd > a+b+c+d$.

7 Let a, b and c be positive real numbers such that $abc = 1$. Prove the inequality $\left(ab+bc+\frac{1}{ca}\right)\left(bc+ca+\frac{1}{ab}\right)\left(ca+ab+\frac{1}{bc}\right) \geq (1+2a)(1+2b)(1+2c)$.

8 Show that $(x+y+z)(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}) \geq 4(\frac{x}{xy+1}+\frac{y}{yz+1}+\frac{z}{zx+1})^2$, for all real positive numbers x, y and z.

9 Consider an integer $n \geq 4$ and a sequence of real numbers $x_1, x_2, x_3, ..., x_n$. An operation consists in eliminating all numbers not having the rank of the form $4k+3$, thus leaving only the numbers $x_3.x_7.x_{11}, ...$(for example, the sequence $4, 5, 9, 3, 6, 6, 1, 8$ produces the sequence $9, 1$).

Upon the sequence $1, 2, 3, ..., 1024$ the operation is performed successively for 5 times. Show that at the end only one number remains and find this number.

– Combinatorics

1 On a 5×5 board, n white markers are positioned, each marker in a distinct 1×1 square. A smart child got an assignment to recolor in black as many markers as possible, in the following manner: a white marker is taken from the board, it is colored in black, and then put back on the board on an empty square such that none of the neighboring squares contains a white marker (two squares are called neighboring if they share a common side).
If it is possible for the child to succeed in coloring all the markers black, we say that the initial positioning of the markers was *good*.
a) Prove that if $n = 20$, then a *good* initial positioning exists.
b) Prove that if $n = 21$, then a *good* initial positioning does not exist.

2 Kostas and Helene have the following dialogue:
Kostas: I have in my mind three positive real numbers with product 1 and sum equal to the sum of all their pairwise products.
Helene: I think that I know the numbers you have in mind. They are all equal to 1.
Kostas: In fact, the numbers you mentioned satisfy my conditions, but I did not think of these numbers. The numbers you mentioned have the minimal sum between all possible solutions of the problem.
Can you decide if Kostas is right? (Explain your answer).

3 Integers $1, 2, ..., 2n$ are arbitrarily assigned to boxes labeled with numbers $1, 2, ..., 2n$. Now, we add the number assigned to the box to the number on the box label. Show that two such sums give the same remainder modulo $2n$.

4 Every cell of table 4×4 is colored into white. It is permitted to place the cross (pictured below) on the table such that its center lies on the table (the whole figure does not need to lie on the table) and change colors of every cell which is covered into opposite (white and black). Find all n such that after n steps it is possible to get the table with every cell colored black.

– Geometry

1 Two perpendicular chords of a circle, AM, BN , which intersect at point K, define on the circle four arcs with pairwise different length, with AB being the smallest of them. We draw the chords AD, BC with $AD//BC$ and C, D different from N, M . If L is the intersection point of DN, MC and T the intersection point of DC, KL, prove that $\angle KTC = \angle KNL$.

2 For a fixed triangle ABC we choose a point M on the ray CA (after A), a point N on the ray AB (after B) and a point P on the ray BC (after C) in a way such that $AM - BC = BN - AC = CP – AB$. Prove that the angles of triangle MNP do not depend on the choice of M, N, P .

3 The vertices A and B of an equilateral triangle ABC lie on a circle k of radius 1, and the vertex C is in the interior of the circle k. A point D, different from B, lies on k so that $AD = AB$. The line DC intersects k for the second time at point E. Find the length of the line segment CE.

4 Let ABC be a triangle, ($BC < AB$). The line l passing trough the vertices C and orthogonal to the angle bisector BE of $\angle B$, meets BE and the median BD of the side AC at points F and G, respectively. Prove that segment DF bisects the segment EG.

5 Is it possible to cover a given square with a few congruent right-angled triangles with acute angle equal to 30^o? (The triangles may not overlap and may not exceed the margins of the square.)

6 Let ABC be a triangle with $\angle A < 90^o$. Outside of a triangle we consider isosceles triangles ABE and ACZ with bases AB and AC, respectively. If the midpoint D of the side BC is such that $DE \perp DZ$ and $EZ = 2 \cdot ED$, prove that $\angle AEB = 2 \cdot \angle AZC$.

7 Let ABC be an isosceles triangle with $AC = BC$. The point D lies on the side AB such that the semicircle with diameter BD and center O is tangent to the side AC in the point P and intersects the side BC at the point Q. The radius OP intersects the chord DQ at the point E such that $5 \cdot PE = 3 \cdot DE$. Find the ratio $\frac{AB}{BC}$.

8 The side lengths of a parallelogram are a, b and diagonals have lengths x and y. Knowing that $ab = \frac{xy}{2}$, show that $(a, b) = \left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{2}}\right)$ or $(a, b) = \left(\frac{y}{\sqrt{2}}, \frac{x}{\sqrt{2}}\right)$.

9 Let O be a point inside the parallelogram $ABCD$ such that $\angle AOB + \angle COD = \angle BOC + \angle AOD$. Prove that there exists a circle k tangent to the circumscribed circles of the triangles $\triangle\ AOB, \triangle\ BOC, \triangle\ COD$ and $\triangle\ DOA$.

10 Let Γ be a circle of center O, and δ. be a line in the plane of Γ, not intersecting it. Denote by A the foot of the perpendicular from O onto δ., and let M be a (variable) point on Γ. Denote by γ the circle of diameter AM , by X the (other than M) intersection point of γ and Γ, and by Y the (other than A) intersection point of γ and δ. Prove that the line XY passes through a fixed point.

11 Consider ABC an acute-angled triangle with $AB \neq AC$. Denote by M the midpoint of BC, by D, E the feet of the altitudes from B, C respectively and let P be the intersection point of the lines DE and BC. The perpendicular from M to AC meets the perpendicular from C to BC at point R. Prove that lines PR and AM are perpendicular.

– Number Theory

1 Find all the positive integers x and y that satisfy the equation $x(x - y) = 8y - 7$

2 Let $n \geq 2$ be a fixed positive integer. An integer will be called "n-free" if it is not a multiple of an n-th power of a prime. Let M be an infi nite set of rational numbers, such that the product of every n elements of M is an n-free integer. Prove that M contains only integers.

3 Let $s(a)$ denote the sum of digits of a given positive integer a. The sequence $a_1, a_2, ..., a_n, ...$ of positive integers is such that $a_{n+1} = a_n + s(a_n)$ for each positive integer n. Find the greatest possible n for which it is possible to have $a_n = 2008$.

4 Find all integers n such that $n^4 + 8n + 11$ is a product of two or more consecutive integers.

5 Is it possible to arrange the numbers $1^1, 2^2, ..., 2008^{2008}$ one after the other, in such a way that the obtained number is a perfect square? (Explain your answer.)

6 Let $f : N \to R$ be a function, satisfying the following condition:
for every integer $n > 1$, there exists a prime divisor p of n such that $f(n) = f\left(\frac{n}{p}\right) - f(p)$.
If $f(2^{2007}) + f(3^{2008}) + f(5^{2009}) = 2006$, determine the value of $f(2007^2) + f(2008^3) + f(2009^5)$

7 Determine the minimum value of prime $p > 3$ for which there is no natural number $n > 0$ such that $2^n + 3^n \equiv 0 \pmod{p}$.

8 Let a, b, c, d, e, f are nonzero digits such that the natural numbers $\overline{abc}$, $\overline{def}$ and $\overline{abcdef}$ are squares.
a) Prove that $\overline{abcdef}$ can be represented in two different ways as a sum of three squares of natural numbers.
b) Give an example of such a number.

9 Let p be a prime number. Find all positive integers a and b such that: $\frac{4a+p}{b} + \frac{4b+p}{a}$ and $\frac{a^2}{b} + \frac{b^2}{a}$ are integers.

10 Prove that $2^n + 3^n$ is not a perfect cube for any positive integer n.

11 Determine the greatest number with n digits in the decimal representation which is divisible by 429 and has the sum of all digits less than or equal to 11.

12 Find all prime numbers p, q, r, such that $\frac{p}{q} - \frac{4}{r+1} = 1$