JBMO Shortlist 2007

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–	Algebra	
1	Let a be positive real number such that $a^3 = 6(a+1)$. Prove that the equation $x^2+ax+a^2-6=0$ has no real solution.	
2	A2 Prove that for all Positive reals a, b, c $\frac{a^2-bc}{2a^2+bc}+\frac{b^2-ca}{2b^2+ca}+\frac{c^2-ab}{2c^2+ab}\leq 0$	
3	Let A be a set of positive integers containing the number 1 and at least one more element. Given that for any two different elements m, n of A the number $\frac{m+1}{(m+1,n+1)}$ is also an element of A, prove that A coincides with the set of positive integers.	
4	Let a and b be positive integers bigger than 2. Prove that there exists a positive integer k and a sequence $n_1, n_2, ..., n_k$ consisting of positive integers, such that $n_1 = a, n_k = b$, and $(n_i + n_{i+1}) \| n_i n_{i+1}$ for all $i = 1, 2, ..., k-1$	
5	The real numbers x, y, z, m, n are positive, such that $m+n \geq 2$. Prove that $x\sqrt{yz(x+my)(x+nz)}+y\sqrt{xz(y+mx)(y+nz)}+z\sqrt{xy(z+mx)(x+ny)} \leq \frac{3(m+n)}{8}(x+y)(y+z)(z+x)$	
–	Combinatorics	
1	We call a tiling of an $m \times n$ rectangle with corners (see figure below) "regular" if there is no sub-rectangle which is tiled with corners. Prove that if for some m and n there exists a "regular" tiling of the $m \times n$ rectangular then there exists a "regular" tiling also for the $2m \times 2n$ rectangle.	
2	Given are 50 points in the plane, no three of them belonging to a same line. Each of these points is colored using one of four given colors. Prove that there is a color and at least 130 scalene triangles with vertices of that color.	
3	The nonnegative integer n and $(2n+1)\times(2n+1)$ chessboard with squares colored alternatively black and white are given. For every natural number m with $1 < m < 2n+1$, an $m \times m$ square of the given chessboard that has more than half of its area colored in black, is called a B-square. If the given chessboard is a B-square, fi nd in terms of n the total number of B-squares of this chessboard.	
–	Geometry	

1 G1 Let M be interior point of the triangle ABC with ¡BAC=70and ¡ABC=80 If ¡ACM=10 and ¡CBM=20.Prove that $AB = MC$

2 Let $ABCD$ be a convex quadrilateral with $\angle DAC = \angle BDC = 36^\circ$, $\angle CBD = 18^\circ$ and $\angle BAC = 72^\circ$. The diagonals and intersect at point P . Determine the measure of $\angle APD$.

3 Let the inscribed circle of the triangle $\triangle ABC$ touch side BC at M , side CA at N and side AB at P . Let D be a point from $[NP]$ such that $\frac{DP}{DN} = \frac{BD}{CD}$. Show that $DM \perp PN$.

4 Let S be a point inside $\angle pOq$, and let k be a circle which contains S and touches the legs Op and Oq in points P and Q respectively. Straight line s parallel to Op from S intersects Oq in a point R. Let T be the intersection point of the ray PS and circumscribed circle of $\triangle SQR$ and $T \neq S$. Prove that $OT//SQ$ and OT is a tangent of the circumscribed circle of $\triangle SQR$.

– Number Theory

1 Find all the pairs positive integers (x, y) such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{[x,y]} + \frac{1}{(x,y)} = \frac{1}{2}$, where (x, y) is the greatest common divisor of x, y and $[x, y]$ is the least common multiple of x, y.

2 Prove that the equation $x^{2006} - 4y^{2006} - 2006 = 4y^{2007} + 2007y$ has no solution in the set of the positive integers.

3 Let $n > 1$ be a positive integer and p a prime number such that $n|(p-1)$and $p|(n^6 - 1)$. Prove that at least one of the numbers $p - n$ and $p + n$ is a perfect square.

4 Let a, b be two co-prime positive integers. A number is called *good* if it can be written in the form $ax + by$ for non-negative integers x, y. Defi ne the function $f : Z \to Z$as $f(n) = n - n_a - n_b$, where s_t represents the remainder of s upon division by t. Show that an integer n is *good* if and only if the in finite sequence $n, f(n), f(f(n)), ...$ contains only non-negative integers.

5 Prove that if p is a prime number, then $7p + 3^p - 4$ is not a perfect square.

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