Romania Team Selection Tests 2016

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by MarkBcc168, ThE-dArK-lOrD, va2010, ABCDE, CantonMathGuy

– **Day 1**

1 Two circles, ω_1 and ω_2, centered at O_1 and O_2, respectively, meet at points A and B. A line through B meet ω_1 again at C, and ω_2 again at D. The tangents to ω_1 and ω_2 at C and D, respectively, meet at E, and the line AE meets the circle ω through A, O_1, O_2 again at F. Prove that the length of the segment EF is equal to the diameter of ω.

2 Let n be a positive integer, and let $S_1, S_2, , S_n$ be a collection of finite non-empty sets such that

$$\sum_{1\le i<j\le n} \frac{|S_i \cap S_j|}{|S_i||S_j|} < 1.$$

Prove that there exist pairwise distinct elements $x_1, x_2, , x_n$ such that x_i is a member of S_i for each index i.

3 Let n be a positive integer, and let $a_1, a_2, .., a_n$ be pairwise distinct positive integers. Show that

$$\sum_{k=1}^{n} \frac{1}{[a_1, a_2, , a_k]} < 4,$$

where $[a_1, a_2, , a_k]$ is the least common multiple of the integers $a_1, a_2, , a_k$.

4 Determine the integers $k \ge 2$ for which the sequence $\left\{\binom{2n}{n} \pmod{k}\right\}_{n\in\mathbb{Z}_{\ge 0}}$ is eventually periodic.

– **Day 2**

1 Given positive integers k and m, show that m and $\binom{n}{k}$ are coprime for infinitely many integers $n \ge k$.

2 Let ABC be an acute triangle and let M be the midpoint of AC. A circle ω passing through B and M meets the sides AB and BC at points P and Q respectively. Let T be the point such that $BPTQ$ is a parallelogram. Suppose that T lies on the circumcircle of ABC. Determine all possible values of $\frac{BT}{BM}$.

3 Prove that:
(a) If $(a_n)_{n\ge 1}$ is a strictly increasing sequence of positive integers such that $\frac{a_{2n-1}+a_{2n}}{a_n}$ is a constant as n runs through all positive integers, then this constant is an integer greater than or

equal to 4; and
(b) Given an integer $N \geq 4$, there exists a strictly increasing sequene $(a_n)_{n\geq 1}$ of positive integers such that $\frac{a_{2n-1}+a_{2n}}{a_n} = N$ for all indices n.

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4 Given any positive integer n, prove that:
(a) Every n points in the closed unit square $[0,1] \times [0,1]$ can be joined by a path of length less than $2\sqrt{n}+4$; and
(b) There exist n points in the closed unit square $[0,1] \times [0,1]$ that cannot be joined by a path of length less than $\sqrt{n}-1$.

– **Day 3**

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1 Given a positive integer n, determine all functions f from the first n positive integers to the positive integers, satisfying the following two conditions: **(1)** $\sum_{k=1}^{n} f(k) = 2n$; and **(2)** $\sum_{k\in K} f(k) = n$ for no subset K of the first n positive integers.

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2 Given a positive integer k and an integer $a \equiv 3 \pmod 8$, show that $a^m + a + 2$ is divisible by 2^k for some positive integer m.

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3 Given a positive integer n, show that for no set of integers modulo n, whose size exceeds $1+\sqrt{n+4}$, is it possible that the pairwise sums of unordered pairs be all distinct.

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4 Let $ABCD$ be a convex quadrilateral, and let P, Q, R, and S be points on the sides AB, BC, CD, and DA, respectively. Let the line segment PR and QS meet at O. Suppose that each of the quadrilaterals $APOS$, $BQOP$, $CROQ$, and $DSOR$ has an incircle. Prove that the lines AC, PQ, and RS are either concurrent or parallel to each other.

– **Day 4**

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1 Determine the planar finite configurations C consisting of at least 3 points, satisfying the following conditions; if x and y are distinct points of C, there exist $z \in C$ such that xyz are three vertices of equilateral triangles

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2 Let ABC be a triangle with $CA \neq CB$. Let D, F, and G be the midpoints of the sides AB, AC, and BC respectively. A circle Γ passing through C and tangent to AB at D meets the segments AF and BG at H and I, respectively. The points H' and I' are symmetric to H and I about F and G, respectively. The line $H'I'$ meets CD and FG at Q and M, respectively. The line CM meets Γ again at P. Prove that $CQ = QP$.

Proposed by El Salvador

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3 Given a prime p, prove that the sum $\sum_{k=1}^{\lfloor \frac{q}{p} \rfloor} k^{p-1}$ is not divisible by q for all but finitely many primes q.

–	**Day 5**
1	Determine the positive integers expressible in the form $\frac{x^2+y}{xy+1}$, for at least 2 pairs (x,y) of positive integers
2	Determine all $f:\mathbb{Z}^+\to\mathbb{Z}^+$ such that $f(m)\geq m$ and $f(m+n)\mid f(m)+f(n)$ for all $m,n\in\mathbb{Z}^+$
3	A set $S=\{s_1,s_2,...,s_k\}$ of positive real numbers is "polygonal" if $k\geq 3$ and there is a non-degenerate planar $k-$gon whose side lengths are exactly $s_1,s_2,...,s_k$; the set S is multipolygonal if in every partition of S into two subsets,each of which has at least three elements, exactly one of these two subsets in polygonal. Fix an integer $n\geq 7$. (a) Does there exist an $n-$element multipolygonal set, removal of whose maximal element leaves a multipolygonal set? (b) Is it possible that every $(n-1)-$element subset of an $n-$element set of positive real numbers be multipolygonal?