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– Paper 1

1 Let p_1, p_2, p_3 and p_4 be four different prime numbers satisying the equations

$2p_1 + 3p_2 + 5p_3 + 7p_4 = 162$
$11p_1 + 7p_2 + 5p_3 + 4p_4 = 162$

Find all possible values of the product $p_1p_2p_3p_4$

2 For positive real numbers a, b, c and d such that $a^2 + b^2 + c^2 + d^2 = 1$ prove that
$a^2b^2cd + +ab^2c^2d + abc^2d^2 + a^2bcd^2 + a^2bc^2d + ab^2cd^2 \leq 3/32$,

and determine the cases of equality.

3 Determine, with proof, all integers x for which $x(x + 1)(x + 7)(x + 8)$ is a perfect square.

4 How many sequences $a_1, a_2, ..., a_{2008}$ are there such that each of the numbers $1, 2, ..., 2008$ occurs once in the sequence, and $i \in (a_1, a_2, ..., a_i)$ for each i such that $2 \leq i \leq 2008$?

5 A triangle ABC has an obtuse angle at B. The perpindicular at B to AB meets AC at D, and $|CD| = |AB|$.
Prove that $|AD|^2 = |AB|.|BC|$ if and only if $\angle CBD = 30^\circ$.

– Paper 2

1 Find, with proof, all triples of integers (a, b, c) such that a, b and c are the lengths of the sides of a right angled triangle whose area is $a + b + c$

2 Circles S and T intersect at P and Q, with S passing through the centre of T. Distinct points A and B lie on S, inside T, and are equidistant from the centre of T. The line PA meets T again at D. Prove that $|AD| = |PB|$.

3 Find $a_3, a_4, ..., a_{2008}$, such that $a_i = \pm 1$ for $i = 3, ..., 2008$ and
$\sum\limits_{i=3}^{2008} a_i 2^i = 2008$
and show that the numbers $a_3, a_4, ..., a_{2008}$ are uniquely determined by these conditions.

4 Given $k \in [0,1,2,3]$ and a positive integer n, let $f_k(n)$ be the number of sequences $x_1, ..., x_n$, where $x_i \in [-1,0,1]$ for $i = 1, ..., n$, and
$x_1 + ... + x_n \equiv k$ mod 4

a) Prove that $f_1(n) = f_3(n)$ for all positive integers n.

(b) Prove that
$f_0(n) = [3^n + 2 + [-1]^n]/4$

for all positive integers n.

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5 Suppose that x, y and z are positive real numbers such that $xyz \geq 1$.

(a) Prove that $27 \leq (1+x+y)^2 + (1+x+z)^2 + (1+y+z)^2$,
with equality if and only if $x = y = z = 1$.

(b) Prove that $(1+x+y)^2 + (1+x+z)^2 + (1+y+z)^2 \leq 3(x+y+z)^2$,
with equality if and only if $x = y = z = 1$.
