China Team Selection Test 2018

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Test 1 Day 1

1 Let p, q be positive reals with sum 1. Show that for any n-tuple of reals $(y_1, y_2, ..., y_n)$, there exists an n-tuple of reals $(x_1, x_2, ..., x_n)$ satisfying

$$p \cdot \max\{x_i, x_{i+1}\} + q \cdot \min\{x_i, x_{i+1}\} = y_i$$

for all $i = 1, 2, ..., 2017$, where $x_{2018} = x_1$.

2 A number n is *interesting* if 2018 divides $d(n)$ (the number of positive divisors of n). Determine all positive integers k such that there exists an infinite arithmetic progression with common difference k whose terms are all interesting.

3 Circle ω is tangent to sides AB,AC of triangle ABC at D,E respectively, such that $D \neq B$, $E \neq C$ and $BD + CE < BC$. F,G lies on BC such that $BF = BD$, $CG = CE$. Let DG and EF meet at K. L lies on minor arc DE of ω, such that the tangent of L to ω is parallel to BC. Prove that the incenter of $\triangle ABC$ lies on KL.

Test 1 Day 2

4 Functions $f, g : \mathbb{Z} \to \mathbb{Z}$ satisfy

$$f(g(x) + y) = g(f(y) + x)$$

for any integers x, y. If f is bounded, prove that g is periodic.

5 Given a positive integer k, call n *good* if among

$$\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, ..., \binom{n}{n}$$

at least $0.99n$ of them are divisible by k. Show that exists some positive integer N such that among $1, 2, ..., N$, there are at least $0.99N$ good numbers.

6 Let A_1, A_2, $\cdots$, A_m be m subsets of a set of size n. Prove that

$$\sum_{i=1}^{m} \sum_{j=1}^{m} |A_i| \cdot |A_i \cap A_j| \geq \frac{1}{mn} \left(\sum_{i=1}^{m} |A_i| \right)^3.$$

Test 2 Day 1

1 Given a triangle ABC. D is a moving point on the edge BC. Point E and Point F are on the edge AB and AC, respectively, such that $BE=CD$ and $CF=BD$. The circumcircle of $\triangle BDE$ and $\triangle CDF$ intersects at another point P other than D. Prove that there exists a fixed point Q, such that the length of QP is constant.

2 An integer partition, is a way of writing n as a sum of positive integers. Two sums that differ only in the order of their summands are considered the same partition.

For example, 4 can be partitioned in five distinct ways:
4
3 + 1
2 + 2
2 + 1 + 1
1 + 1 + 1 + 1

The number of partitions of n is given by the partition function $p\left (n \right)$. So $p\left (4 \right) = 5$.
Determine all the positive integers so that $p\left (n \right)+p\left (n+4 \right)=p\left (n+2 \right)+p\left (n+3 \right)$.

3 Two positive integers $p,q \in \mathbf{Z}^{+}$ are given. There is a blackboard with n positive integers written on it. A operation is to choose two same number a,a written on the blackboard, and replace them with $a+p,a+q$. Determine the smallest n so that such operation can go on infinitely.

Test 2 Day 2

4 Let k, M be positive integers such that $k-1$ is not squarefree. Prove that there exist a positive real α, such that $\lfloor \alpha\cdot k^n \rfloor$ and M are coprime for any positive integer n.

5 Given positive integers n, k such that $n\ge 4k$, find the minimal value $\lambda=\lambda(n,k)$ such that for any positive reals $a_1,a_2,\ldots,a_n$, we have

$$\sum\limits_{i=1}^{n} {\frac{{a}_{i}}{\sqrt{{a}_{i}^{2}+{a}_{{i}+{1}}^{2}+{\cdots}{{+}}{a}_{{i}{+}{k}}^{2}}}} \le \lambda$$

Where $a_{n+i}=a_i,i=1,2,\ldots,k$

6 Let M,a,b,r be non-negative integers with $a,r\ge 2$, and suppose there exists a function $f:\mathbb{Z}\rightarrow\mathbb{Z}$ satisfying the following conditions:
(1) For all $n\in \mathbb{Z}$, $f^{(r)}(n)=an+b$ where $f^{(r)}$ denotes the composition of r copies of f
(2) For all $n\ge M$, $f(n)\ge 0$

(3) For all $n > m > M$, $n - m | f(n) - f(m)$
Show that a is a perfect r-th power.

Test3 Day 1

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1 Let ω_1, ω_2 be two non-intersecting circles, with circumcenters O_1, O_2 respectively, and radii r_1, r_2 respectively where $r_1 < r_2$. Let AB, XY be the two internal common tangents of ω_1, ω_2, where A, X lie on ω_1, B, Y lie on ω_2. The circle with diameter AB meets ω_1, ω_2 at P and Q respectively. If

$$\angle AO_1P + \angle BO_2Q = 180^\circ,$$

find the value of $\frac{PX}{QY}$ (in terms of r_1, r_2).

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2 Let G be a simple graph with 100 vertices such that for each vertice u, there exists a vertice $v \in N(u)$ and $N(u) \cap N(v) = \emptyset$. Try to find the maximal possible number of edges in G. The $N(.)$ refers to the neighborhood.

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3 Prove that there exists a constant $C > 0$ such that

$$H(a_1) + H(a_2) + \cdots + H(a_m) \leq C \sqrt{\sum_{i=1}^{m} i a_i}$$

holds for arbitrary positive integer m and any m positive integer $a_1, a_2, \cdots, a_m$, where

$$H(n) = \sum_{k=1}^{n} \frac{1}{k}.$$

Test 3 Day 2

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4 Suppose $A_1, A_2, \cdots, A_n \subseteq \{1, 2, \cdots, 2018\}$ and $|A_i| = 2, i = 1, 2, \cdots, n$, satisfying that

$$A_i + A_j, \ 1 \leq i \leq j \leq n,$$

are distinct from each other. $A + B = \{a + b | a \in A, \ b \in B\}$. Determine the maximal value of n.

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5 Let ABC be a triangle with $\angle BAC > 90^\circ$, and let O be its circumcenter and ω be its circumcircle. The tangent line of ω at A intersects the tangent line of ω at B and C respectively at point P and Q. Let D, E be the feet of the altitudes from P, Q onto BC, respectively. F, G are two points on $\overline{PQ}$ different from A, so that A, F, B, E and A, G, C, D are both concyclic. Let M be the midpoint of $\overline{DE}$. Prove that DF, OM, EG are concurrent.

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6 Find all pairs of positive integers (x, y) such that $(xy+1)(xy+x+2)$ be a perfect square .

Test 4 Day 1

1 Define the polymonial sequence $\left\{f_n\left(x\right)\right\}_{n\ge 1}$ with $f_1\left(x\right)=1$,

$$f_{2n}\left(x\right)=xf_n\left(x\right),\; f_{2n+1}\left(x\right) = f_n\left(x\right)+ f_{n+1}\left(x\right),\; n\ge 1.$$

Look for all the rational number a which is a root of certain $f_n\left(x\right).$

2 There are 32 students in the class with 10 interesting group. Each group contains exactly 16 students. For each couple of students, the square of the number of the groups which are only involved by just one of the two students is defined as their $interests-disparity$. Define S as the sum of the $interests-disparity$ of all the couples, $\binom{32}{2}\left (=\text{496} \right)$ ones in total. Determine the minimal possible value of S.

3 In isosceles $\triangle ABC$, $AB=AC$, points D,E,F lie on segments BC,AC,AB such that $DE\parallel AB$, $DF\parallel AC$. The circumcircle of $\triangle ABC$ ω_1 and the circumcircle of $\triangle AEF$ ω_2 intersect at A,G. Let DE meet ω_2 at $K\neq E$. Points L,M lie on ω_1,ω_2 respectively such that $LG\perp KG, MG\perp CG$. Let P,Q be the circumcenters of $\triangle DGL$ and $\triangle DGM$ respectively. Prove that A,G,P,Q are concyclic.

Test 4 Day 2

4 Let p be a prime and k be a positive integer. Set S contains all positive integers a satisfying $1\le a \le p-1$, and there exists positive integer x such that $x^k\equiv a \pmod p$.
Suppose that $3\le |S| \le p-2$. Prove that the elements of S, when arranged in increasing order, does not form an arithmetic progression.

5 Suppose the real number $\lambda \in \left(0,1\right),$ and let n be a positive integer. Prove that the modulus of all the roots of the polynomial

$$f\left (x \right)=\sum_{k=0}^{n}\binom{n}{k}\lambda^{k\left (n-k \right)}x^{k}$$

are $1.$

6 Suppose $a_i, b_i, c_i, i=1,2,\cdots ,n$, are $3n$ real numbers in the interval $\left [0,1 \right].$ Define

$$S=\left \{ \left (i,j,k \right) |\, a_i+b_j+c_k<1 \right \}, \; T=\left \{ \left (i,j,k \right) |\, a_i+b_j+c_k>2 \right \}.$$

Now we know that $\left | S \right |\ge 2018,\, \left | T \right |\ge 2018.$ Try to find the minimal possible value of n.