AoPS Online
Art of Problem Solving

AMC 12/AHSME 2018

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– A

– February 7th, 2018

1 A large urn contains 100 balls, of which 36% are red and the rest are blue. How many of the blue balls must be removed so that the percentage of red balls in the urn will be 72%? (No red balls are to be removed.)

(A) 28 **(B)** 32 **(C)** 36 **(D)** 50 **(E)** 64

2 While exploring a cave, Carl comes across a collection of 5-pound rocks worth $\$14$ each, 4-pound rocks worth $\$11$ each, and 1-pound rocks worth $\$2$ each. There are at least 20 of each size. He can carry at most 18 pounds. What is the maximum value, in dollars, of the rocks he can carry out of the cave?

(A) 48 **(B)** 49 **(C)** 50 **(D)** 51 **(E)** 52

3 How many ways can a student schedule 3 mathematics courses – algebra, geometry, and number theory – in a 6-period day if no two mathematics courses can be taken in consecutive periods? (What courses the student takes during the other 3 periods is of no concern here.)

(A) 3 **(B)** 6 **(C)** 12 **(D)** 18 **(E)** 24

4 Alice, Bob, and Charlie were on a hike and were wondering how far away the nearest town was. When Alice said, "We are at least 6 miles away," Bob replied, "We are at most 5 miles away." Charlie then remarked, "Actually the nearest town is at most 4 miles away." It turned out that none of the three statements were true. Let d be the distance in miles to the nearest town. Which of the following intervals is the set of all possible values of d?

(A) $(0,4)$ **(B)** $(4,5)$ **(C)** $(4,6)$ **(D)** $(5,6)$ **(E)** $(5,\infty)$

5 What is the sum of all possible values of k for which the polynomials x^2-3x+2 and x^2-5x+k have a root in common?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 10

6 For positive integers m and n such that $m+10<n+1$, both the mean and the median of the set $\{m, m+4, m+10, n+1, n+2, 2n\}$ are equal to n. What is $m+n$?

(A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

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7 For how many (not necessarily positive) integer values of n is the value of $4000\cdot\left(\frac{2}{5}\right)^n$ an integer?

(A) 3 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** 9

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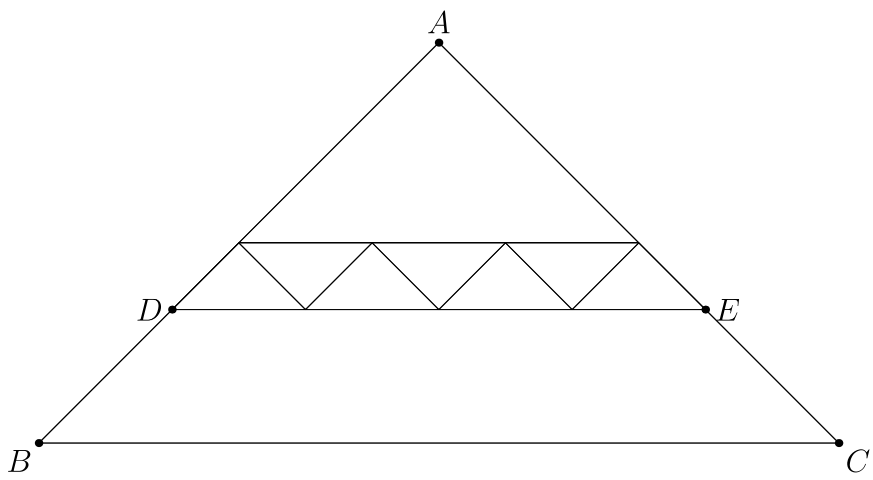
8 All of the triangles in the diagram below are similar to iscoceles triangle ABC, in which $AB = AC$. Each of the 7 smallest triangles has area 1, and $\triangle ABC$ has area 40. What is the area of trapezoid $DBCE$?



(A) 16 **(B)** 18 **(C)** 20 **(D)** 22 **(E)** 24

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9 Which of the following describes the largest subset of values of y within the closed interval $[0,\pi]$ for which

$$\sin(x+y) \leq \sin(x) + \sin(y)$$

for every x between 0 and π, inclusive? **(A)** $y=0$ **(B)** $0 \leq y \leq \frac{\pi}{4}$ **(C)** $0 \leq y \leq \frac{\pi}{2}$ **(D)** $0 \leq y \leq \frac{3\pi}{4}$ **(E)** $0 \leq y \leq \pi$

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10 How many ordered pairs of real numbers (x, y) satisfy the following system of equations?

$$x + 3y = 3$$
$$\big||x| - |y|\big| = 1$$

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 8

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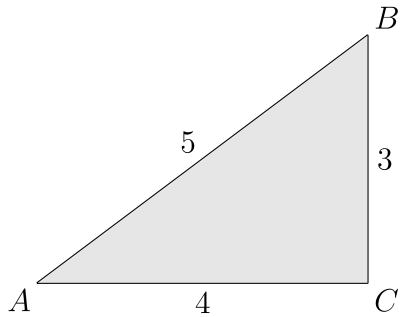
11 A paper triangle with sides of lengths 3, 4, and 5 inches, as shown, is folded so that point A falls on point B. What is the length in inches of the crease?



(A) $1 + \frac{1}{2}\sqrt{2}$ **(B)** $\sqrt{3}$ **(C)** $\frac{7}{4}$ **(D)** $\frac{15}{8}$ **(E)** 2

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12 Let S be a set of 6 integers taken from $\{1, 2, \dots, 12\}$ with the property that if a and b are elements of S with $a < b$, then b is not a multiple of a. What is the least possible value of an element in S?

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 7

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13 How many nonnegative integers can be written in the form

$$a_7 \cdot 3^7 + a_6 \cdot 3^6 + a_5 \cdot 3^5 + a_4 \cdot 3^4 + a_3 \cdot 3^3 + a_2 \cdot 3^2 + a_1 \cdot 3^1 + a_0 \cdot 3^0,$$

where $a_i \in \{-1, 0, 1\}$ for $0 \le i \le 7$?

(A) 512 **(B)** 729 **(C)** 1094 **(D)** 3281 **(E)** $59,048$

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14 The solution to the equation $\log_{3x} 4 = \log_{2x} 8$, where x is a positive real number other than $\frac{1}{3}$ or $\frac{1}{2}$, can be written as $\frac{p}{q}$ where p and q are relatively prime positive integers. What is $p + q$?

(A) 5 **(B)** 13 **(C)** 17 **(D)** 31 **(E)** 35

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15 A scanning code consists of a 7×7 grid of squares, with some of its squares colored black and the rest colored white. There must be at least one square of each color in this grid of 49 squares. A scanning code is called *symmetric* if its look does not change when the entire square is rotated by a multiple of 90° counterclockwise around its center, nor when it is reflected across a line joining opposite corners or a line joining midpoints of opposite sides. What is the total number of possible symmetric scanning codes?

(A) 510 **(B)** 1022 **(C)** 8190 **(D)** 8192 **(E)** 65,534

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16 Which of the following describes the set of values of a for which the curves $x^2 + y^2 = a^2$ and $y = x^2 - a$ in the real xy-plane intersect at exactly 3 points?

(A) $a = \frac{1}{4}$ **(B)** $\frac{1}{4} < a < \frac{1}{2}$ **(C)** $a > \frac{1}{4}$ **(D)** $a = \frac{1}{2}$ **(E)** $a > \frac{1}{2}$

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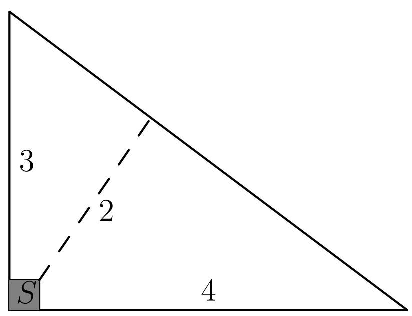
17 Farmer Pythagoras has a field in the shape of a right triangle. The right triangle's legs have lengths 3 and 4 units. In the corner where those sides meet at a right angle, he leaves a small unplanted square S so that from the air it looks like the right angle symbol. The rest of the field is planted. The shortest distance from S to the hypotenuse is 2 units. What fraction of the field is planted?



(A) $\frac{25}{27}$ **(B)** $\frac{26}{27}$ **(C)** $\frac{73}{75}$ **(D)** $\frac{145}{147}$ **(E)** $\frac{74}{75}$

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18 Triangle ABC with $AB = 50$ and $AC = 10$ has area 120. Let D be the midpoint of $\overline{AB}$, and let E be the midpoint of $\overline{AC}$. The angle bisector of $\angle BAC$ intersects $\overline{DE}$ and $\overline{BC}$ at F and G, respectively. What is the area of quadrilateral $FDBG$?

(A) 60 **(B)** 65 **(C)** 70 **(D)** 75 **(E)** 80

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19 Let A be the set of positive integers that have no prime factors other than 2, 3, or 5. The infinite sum

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{12} + \frac{1}{15} + \frac{1}{16} + \frac{1}{18} + \frac{1}{20} + \cdots$$

of the reciprocals of the elements of A can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. What is $m + n$?

(A) 16 **(B)** 17 **(C)** 19 **(D)** 23 **(E)** 36

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20 Triangle ABC is an isosceles right triangle with $AB = AC = 3$. Let M be the midpoint of hypotenuse $\overline{BC}$. Points I and E lie on sides $\overline{AC}$ and $\overline{AB}$, respectively, so that $AI > AE$ and $AIME$ is a cyclic quadrilateral. Given that triangle EMI has area 2, the length CI can be written as $\frac{a-\sqrt{b}}{c}$, where a, b, and c are positive integers and b is not divisible by the square of any prime. What is the value of $a + b + c$?

(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 13

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21 Which of the following polynomials has the greatest real root?

(A) $x^{19}+2018x^{11}+1$ **(B)** $x^{17}+2018x^{11}+1$ **(C)** $x^{19}+2018x^{13}+1$ **(D)** $x^{17}+2018x^{13}+1$ **(E)** $2019x+2018$

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22 The solutions to the equations $z^2=4+4\sqrt{15}i$ and $z^2=2+2\sqrt{3}i$, where $i=\sqrt{-1}$, form the vertices of a parallelogram in the complex plane. The area of this parallelogram can be written in the form $p\sqrt{q}-r\sqrt{s}$, where p, q, r, and s are positive integers and neither q nor s is divisible by the square of any prime number. What is $p+q+r+s$?

(A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 24

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23 In $\triangle PAT$, $\angle P=36^\circ$, $\angle A=56^\circ$, and $PA=10$. Points U and G lie on sides $\overline{TP}$ and $\overline{TA}$, respectively, so that $PU=AG=1$. Let M and N be the midpoints of segments $\overline{PA}$ and $\overline{UG}$, respectively. What is the degree measure of the acute angle formed by lines MN and PA?

(A) 76 **(B)** 77 **(C)** 78 **(D)** 79 **(E)** 80

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24 Alice, Bob, and Carol play a game in which each of them chooses a real number between 0 and 1. The winner of the game is the one whose number is between the numbers chosen by the other two players. Alice announces that she will choose her number uniformly at random from all the numbers between 0 and 1, and Bob announces that he will choose his number uniformly at random from all the numbers between $\frac{1}{2}$ and $\frac{2}{3}$. Armed with this information, what number should Carol choose to maximize her chance of winning?

(A) $\frac{1}{2}$ **(B)** $\frac{13}{24}$ **(C)** $\frac{7}{12}$ **(D)** $\frac{5}{8}$ **(E)** $\frac{2}{3}$

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25 For a positive integer n and nonzero digits a, b, and c, let A_n be the n-digit integer each of whose digits is equal to a; let B_n be the n-digit integer each of whose digits is equal to b, and let C_n be the $2n$-digit (not n-digit) integer each of whose digits is equal to c. What is the greatest possible value of $a+b+c$ for which there are at least two values of n such that $C_n-B_n=A_n^2$?

(A) 12 **(B)** 14 **(C)** 16 **(D)** 18 **(E)** 20

– B

– February 15th, 2018

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1 Kate bakes a 20-inch by 18-inch pan of cornbread. The cornbread is cut into pieces that measure 2 inches by 2 inches. How many pieces of cornbread does the pan contain?

(A) 90 **(B)** 100 **(C)** 180 **(D)** 200 **(E)** 360

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2 Sam drove 96 miles in 90 minutes. His average speed during the first 30 minutes was 60 mph (miles per hour), and his average speed during the second 30 minutes was 65 mph. What was his average speed, in mph, during the last 30 minutes?

(A) 64 **(B)** 65 **(C)** 66 **(D)** 67 **(E)** 68

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3 A line with slope 2 intersects a line with slope 6 at the point $(40, 30)$. What is the distance between the x-intercepts of these two lines?

(A) 5 **(B)** 10 **(C)** 20 **(D)** 25 **(E)** 50

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4 A circle has a chord of length 10, and the distance from the center of the circle to the chord is 5. What is the area of the circle?

(A) 25π **(B)** 50π **(C)** 75π **(D)** 100π **(E)** 125π

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5 How many subsets of $\{2, 3, 4, 5, 6, 7, 8, 9\}$ contain at least one prime number?

(A) 128 **(B)** 192 **(C)** 224 **(D)** 240 **(E)** 256

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6 Suppose S cans of soda can be purchased from a vending machine for Q quarters. Which of the following expressions describes the number of cans of soda that can be purchased for D dollars, where 1 dollar is worth 4 quarters?

(A) $\frac{4DQ}{S}$ **(B)** $\frac{4DS}{Q}$ **(C)** $\frac{4Q}{DS}$ **(D)** $\frac{DQ}{4S}$ **(E)** $\frac{DS}{4Q}$

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7 What is the value of

$$\log_3 7 \cdot \log_5 9 \cdot \log_7 11 \cdot \log_9 13 \cdots \log_{21} 25 \cdot \log_{23} 27?$$

(A) 3 **(B)** $3\log_7 23$ **(C)** 6 **(D)** 9 **(E)** 10

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8 Line segment $\overline{AB}$ is a diameter of a circle with $AB = 24$. Point C, not equal to A or B, lies on the circle. As point C moves around the circle, the centroid (center of mass) of $\triangle ABC$ traces out a closed curve missing two points. To the nearest positive integer, what is the area of the region bounded by this curve?

(A) 25 **(B)** 38 **(C)** 50 **(D)** 63 **(E)** 75

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9 What is

$$\sum_{i=1}^{100}\sum_{j=1}^{100}(i+j)?$$

(A) $100,100$ **(B)** $500,500$ **(C)** $505,000$ **(D)** $1,001,000$ **(E)** $1,010,000$

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10 A list of 2018 positive integers has a unique mode, which occurs exactly 10 times. What is the least number of distinct values that can occur in the list?

(A) 202 **(B)** 223 **(C)** 224 **(D)** 225 **(E)** 234

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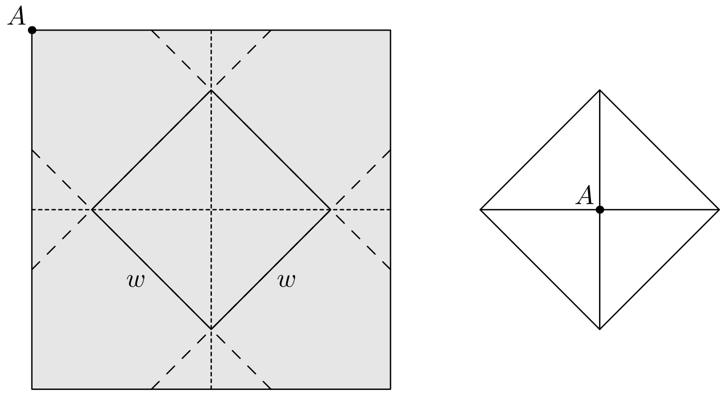
11 A closed box with a square base is to be wrapped with a square sheet of wrapping paper. The box is centered on the wrapping paper with the vertices of the base lying on the midlines of the square sheet of paper, as shown in the figure on the left. The four corners of the wrapping paper are to be folded up over the sides and brought together to meet at the center of the top of the box, point A in the figure on the right. The box has base length w and height h. What is the area of the sheet of wrapping paper?



(A) $2(w+h)^2$ **(B)** $\frac{(w+h)^2}{2}$ **(C)** $2w^2+4wh$ **(D)** $2w^2$ **(E)** w^2h

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12 Side $\overline{AB}$ of $\triangle ABC$ has length 10. The bisector of angle A meets $\overline{BC}$ at D, and $CD = 3$. The set of all possible values of AC is an open interval (m,n). What is $m+n$?

(A) 16 **(B)** 17 **(C)** 18 **(D)** 19 **(E)** 20

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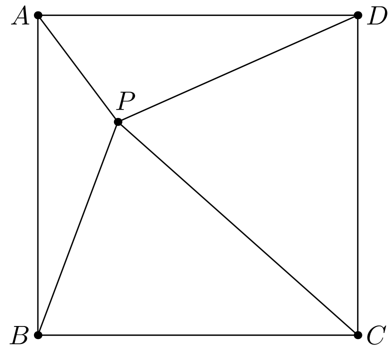
13 Square $ABCD$ has side length 30. Point P lies inside the square so that $AP = 12$ and $BP = 26$. The centroids of $\triangle ABP$, $\triangle BCP$, $\triangle CDP$, and $\triangle DAP$ are the vertices of a convex quadrilateral. What is the area of that quadrilateral?



(A) $100\sqrt{2}$ **(B)** $100\sqrt{3}$ **(C)** 200 **(D)** $200\sqrt{2}$ **(E)** $200\sqrt{3}$

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14 Joey and Chloe and their daughter Zoe all have the same birthday. Joey is 1 year older than Chloe, and Zoe is exactly 1 year old today. Today is the first of the 9 birthdays on which Chloe's age will be an integral multiple of Zoe's age. What will be the sum of the two digits of Joey's age the next time his age is a multiple of Zoe's age?

(A) 7 **(B)** 8 **(C)** 9 **(D)** 10 **(E)** 11

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15 How many odd positive 3-digit integers are divisible by 3 but do not contain the digit 3?

(A) 96 **(B)** 97 **(C)** 98 **(D)** 102 **(E)** 120

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16 The solutions to the equation $(z+6)^8 = 81$ are connected in the complex plane to form a convex regular polygon, three of whose vertices are labeled $A, B,$ and C. What is the least possible area of $\triangle ABC$?

(A) $\frac{1}{6}\sqrt{6}$ **(B)** $\frac{3}{2}\sqrt{2}-\frac{3}{2}$ **(C)** $2\sqrt{3}-3\sqrt{2}$ **(D)** $\frac{1}{2}\sqrt{2}$ **(E)** $\sqrt{3}-1$

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17 Let p and q be positive integers such that

$$\frac{5}{9} < \frac{p}{q} < \frac{4}{7}$$

and q is as small as possible. What is $q-p$?

(A) 7 **(B)** 11 **(C)** 13 **(D)** 17 **(E)** 19

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18 A function f is defined recursively by $f(1) = f(2) = 1$ and

$$f(n) = f(n-1) - f(n-2) + n$$

for all integers $n \geq 3$. What is $f(2018)$?

(A) 2016 **(B)** 2017 **(C)** 2018 **(D)** 2019 **(E)** 2020

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19 Mary chose an even 4-digit number n. She wrote down all the divisors of n in increasing order from left to right: $1, 2, ..., \frac{n}{2}, n$. At some moment Mary wrote 323 as a divisor of n. What is the smallest possible value of the next divisor written to the right of 323?

(A) 324 **(B)** 330 **(C)** 340 **(D)** 361 **(E)** 646

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20 Let $ABCDEF$ be a regular hexagon with side length 1. Denote by X, Y, and Z the midpoints of sides $\overline{AB}, \overline{CD}, \overline{EF}$, respectively. What is the area of the convex hexagon whose interior is the intersection of the interiors of $\triangle ACE$ and $\triangle XYZ$?

(A) $\frac{3}{8}\sqrt{3}$ **(B)** $\frac{7}{16}\sqrt{3}$ **(C)** $\frac{15}{32}\sqrt{3}$ **(D)** $\frac{1}{2}\sqrt{3}$ **(E)** $\frac{9}{16}\sqrt{3}$

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21 In $\triangle ABC$ with side lengths $AB = 13$, $AC = 12$, and $BC = 5$, let O and I denote the circumcenter and incenter, respectively. A circle with center M is tangent to the legs AC and BC and to the circumcircle of $\triangle ABC$. What is the area of $\triangle MOI$?

(A) $5/2$ **(B)** $11/4$ **(C)** 3 **(D)** $13/4$ **(E)** $7/2$

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22 Consider polynomials $P(x)$ of degree at most 3, each of whose coefficients is an element of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. How many such polynomials satisfy $P(-1) = -9$?

(A) 110 **(B)** 143 **(C)** 165 **(D)** 220 **(E)** 286

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23 Ajay is standing at point A near Pontianak, Indonesia, 0° latitude and 110° E longitude. Billy is standing at point B near Big Baldy Mountain, Idaho, USA, 45° N latitude and 115° W longitude. Assume that Earth is a perfect sphere with center C. What is the degree measure of $\angle ACB$?

(A) 105 **(B)** $112\frac{1}{2}$ **(C)** 120 **(D)** 135 **(E)** 150

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24 Let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x. How many real numbers x satisfy the equation $x^2 + 10{,}000\lfloor x \rfloor = 10{,}000x$?

(A) 197 **(B)** 198 **(C)** 199 **(D)** 200 **(E)** 201

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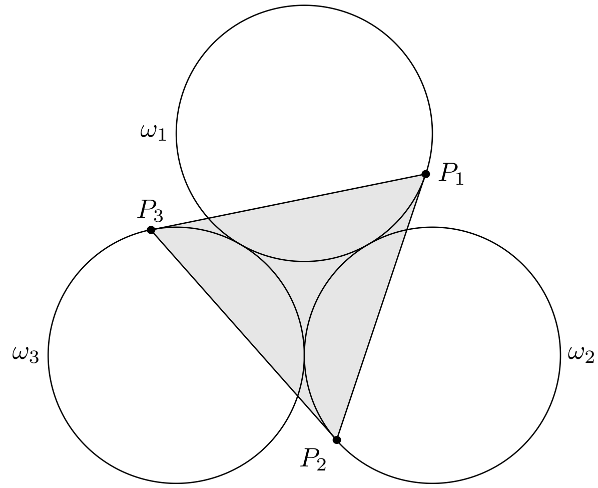
25 Circles ω_1, ω_2, and ω_3 each have radius 4 and are placed in the plane so that each circle is externally tangent to the other two. Points P_1, P_2, and P_3 lie on ω_1, ω_2, and ω_3 respectively such that $P_1P_2 = P_2P_3 = P_3P_1$ and line P_iP_{i+1} is tangent to ω_i for each $i = 1, 2, 3$, where $P_4 = P_1$. See the figure below. The area of $\triangle P_1P_2P_3$ can be written in the form $\sqrt{a} + \sqrt{b}$ for positive integers a and b. What is $a + b$?



(A) 546 **(B)** 548 **(C)** 550 **(D)** 552 **(E)** 554
