Problems from Year 29 USAMTS (2017-2018)

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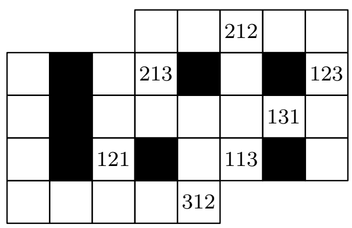
– Round 1

1 Fill each white square in with a number so that each of the 27 three-digit numbers whose digits are all 1, 2, or 3 is used exactly once. For each pair of white squares sharing a side, the two numbers must have equal digits in exactly two of the three positions (ones, tens, hundreds). Some numbers have been given to you.
You do not need to prove that your answer is the only one possible; you merely need to nd an answer that satis es the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justi cation acceptable.)



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2 After each Goober ride, the driver rates the passenger as 1, 2, 3, 4, or 5 stars. The passenger's overall rating is determined as the average of all of the ratings given to him or her by drivers so far. Noah had been on several rides, and his rating was neither 1 nor 5. Then he got a 1 star on a ride because he barfed on the driver. Show that the number of 5 stars that Noah needs in order to climb back to at least his overall rating before bar ng is independent of the number of rides that he had taken.

3 Do there exist two polygons such that, by putting them together in three different ways (without holes, overlap, or reflections), we can obtain first a triangle, then a convex quadrilateral, and lastly a convex pentagon?

4 Two players take turns placing an unused number from
1, 2, 3, 4, 5, 6, 7, 8 into one of the empty squares in the array to the
right. The game ends once all the squares are filled. The first player
wins if the product of the numbers in the top row is greater. The second

player wins if the product of the numbers in the bottom row is greater. If both players play with perfect strategy, who wins this game?

1	2	3	4
5	6	7	8

5 Does there exist a set S consisting of rational numbers with the following property: for every integer n there is a unique nonempty, finite subset of S, whose elements sum to n?

– Round 2

1 Given a rectangular grid with some cells containing one letter, we say a row or column is *edible* if it has more than one cell with a letter and all such cells contain the same letter. Given such a grid, the hungry, hungry letter monster repeats the following procedure: he nds all edible rows and all edible columns and simultaneously eats all the letters in those rows and columns, removing those letters from the grid and leaving those cells empty. He continues this until no more edible rows and columns remain. Call a grid a *meal* if the letter monster can eat all of its letters using this procedure.
In the 7 by 7 grid to the right, ll each empty space with one letter so that the grid is a meal and there are a total of eight Us, nine Ss, ten As, eleven Ms, and eleven Ts. Some letters have been given to you.
You do not need to prove that your answer is the only one possible; you merely need to find an answer that satis fies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justi cation acceptable.)
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2 Let b be a positive integer. Grogg writes down a sequence whose first term is 1. Each term after that is the total number of digits in all the previous terms of the sequence when written in base b. For example, if $b = 3$, the sequence starts $1, 1, 2, 3, 5, 7, 9, 12, \dots$. If $b = 2521$, what is the first positive power of b that does not appear in the sequence?

3 The USAMTS tug-of-war team needs to pick a representative to send to the national tug-of-war convention. They don't care who they send, as long as they don't send the weakest person on the team. Their team consists of 20 people, who each pull with a different constant strength. They want to design a tournament, with each round planned ahead of time, which at the end will allow them to pick a valid representative. Each round of the tournament is a 10-on-10 tug-of-war match. A round may end in one side winning, or in a tie if the strengths of each side are matched.

Show that they can choose a representative using a tournament with 10 rounds.

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4 Zan starts with a rational number $\frac{a}{b}$ written on the board in lowest terms. Then, every second, Zan adds 1 to both the numerator and denominator of the latest fraction and writes the result in lowest terms. Zan stops as soon as he writes a fraction of the form $\frac{n}{n+1}$, for some positive integer n. If $\frac{a}{b}$ started in that form, Zan does nothing.

As an example, if Zan starts with $\frac{13}{19}$, then after one second he writes $\frac{14}{20} = \frac{7}{10}$, then after two seconds $\frac{8}{11}$, then $\frac{9}{12} = \frac{3}{4}$, at which point he stops.

(a) Prove that Zan will stop in less than $b - a$ seconds.

(b) Show that if $\frac{n}{n+1}$ is the final number, then

$$\frac{n-1}{n} < \frac{a}{b} \leq \frac{n}{n+1}.$$

(Proposed by Michael Tang.)

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5 There are n distinct points in the plane, no three of which are collinear. Suppose that A and B are two of these points. We say that segment AB is independent if there is a straight line such that points A and B are on one side of the line, and the other $n-2$ points are on the other side. What is the maximum possible number of independent segments?

– Round 3

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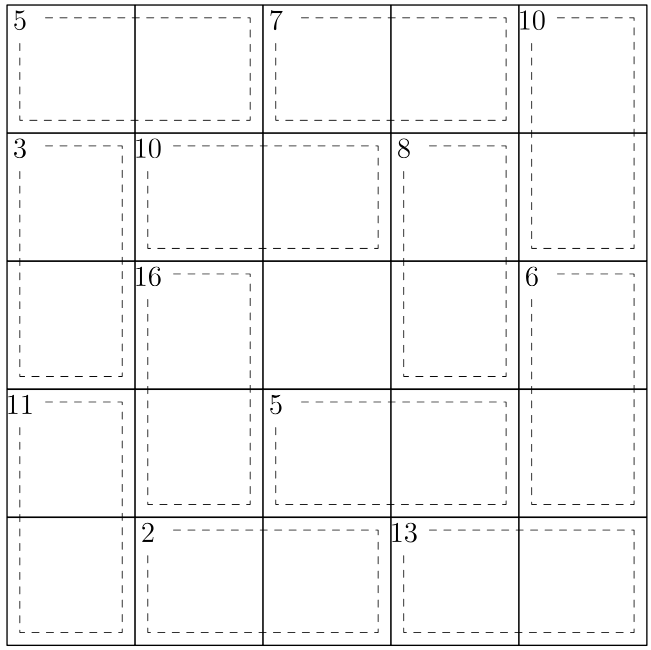
1 Fill in each cell of the grid with a positive digit so that the following conditions hold:
1. each row and column contains ve distinct digits;
2. for any cage containing multiple cells of a row, the label on the cage is the GCD of the sum of the digits in the cage and the sum of the digits in the whole row, and
3. for any cage containing multiple cells of a column, the label on the cage is the GCD of the sum of the digits in the cage and the sum of the digits in the whole column.
You do not need to prove that your answer is the only one possible; you merely need to fi nd an answer that satis fies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justi cation acceptable.)



2 Let q be a real number. Suppose there are three distinct positive integers a, b, c such that $q+a$, $q+b$,$q+c$ is a geometric progression. Show that q is rational.

3 Let ABC be an equilateral triangle with side length 1. Let A_1 and A_2 be the trisection points of AB with A_1 closer to A, B_1 and B_2 be the trisection points of BC with B_1 closer to B, and C_1 and C_2 be the trisection points of CA with C_1 closer to C. Grogg has an orange equilateral triangle the size of triangle $A_1B_1C_1$. He puts the orange triangle over triangle $A_1B_1C_1$ and then rotates it about its center in the shortest direction until its vertices are over $A_2B_2C_2$. Find the area of the region that the orange triangle traveled over during its rotation.

4 A positive integer is called *uphill* if the digits in its decimal representation form an increasing sequence from left to right. That is, a number $\overline{a_1a_2...a_n}$ is uphill if $a_i \leq a_{i+1}$ for all i. For example, 123 and 114 are both uphill. Suppose a polynomial $P(x)$ with rational coefficients takes on an integer value for each uphill positive integer x. Is it necessarily true that $P(x)$ takes on an integer value for each integer x?

5 Let n be a positive integer. Aavid has a card deck consisting of $2n$ cards, each colored with one of n colors such that every color is on exactly two of the cards. The $2n$ cards are randomly ordered in a stack. Every second, he removes the top card from the stack and places the card into an area called the pit. If the other card of that color also happens to be in the pit, Aavid collects both cards of that color and discards them from the pit.
Of the $(2n)!$ possible original orderings of the deck, determine how many have the following property: at every point, the pit contains cards of at most two distinct colors.