Macedonia National Olympiad 2015

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Problem 1 Let AH_A, BH_B and CH_C be altitudes in $\triangle ABC$. Let p_A, p_B, p_C be the perpendicular lines from vertices A, B, C to H_BH_C, H_CH_A, H_AH_B respectively. Prove that p_A, p_B, p_C are concurrent lines.

Problem 2 Let $a, b, c \in \mathbb{R}^+$ such that $abc = 1$. Prove that:

$$a^2b + b^2c + c^2a \geq \sqrt{(a+b+c)(ab+bc+ca)}$$

Problem 3 All contestants at one contest are sitting in n columns and are forming a "good" configuration. (We define one configuration as "good" when we don't have 2 friends sitting in the same column). It's impossible for all the students to sit in $n-1$ columns in a "good" configuration. Prove that we can always choose contestants $M_1, M_2, ..., M_n$ such that M_i is sitting in the $i-th$ column, for each $i = 1, 2, ..., n$ and M_i is friend of M_{i+1} for each $i = 1, 2, ..., n-1$.

Problem 4 Let k_1 and k_2 be two circles and let them cut each other at points A and B. A line through B is cutting k_1 and k_2 in C and D respectively, such that C doesn't lie inside of k_2 and D doesn't lie inside of k_1. Let M be the intersection point of the tangent lines to k_1 and k_2 that are passing through C and D, respectively. Let P be the intersection of the lines AM and CD. The tangent line to k_1 passing through B intersects AD in point L. The tangent line to k_2 passing through B intersects AC in point K. Let $KP \cap MD \equiv N$ and $LP \cap MC \equiv Q$. Prove that $MNPQ$ is a parallelogram.

Problem 5 Find all natural numbers m having exactly three prime divisors p, q, r, such that

$$p-1 \mid m; \quad qr-1 \mid m; \quad q-1 \nmid m; \quad r-1 \nmid m; \quad 3 \nmid q+r.$$
