VMO 2018
www.artofproblemsolving.com/community/c591412
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First day 11th January 2018

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1 The sequence (x_n) is defined as follows:

$$x_1 = 2,\ x_{n+1} = \sqrt{x_n + 8} - \sqrt{x_n + 3}$$

for all $n \geq 1$.
a. Prove that (x_n) has a finite limit and find that limit.
b. For every $n \geq 1$, prove that

$$n \leq x_1 + x_2 + \cdots + x_n \leq n + 1.$$

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2 We have a scalene acute triangle ABC (triangle with no two equal sides) and a point D on side BC. Pick a point E on side AB and a point F on side AC such that $\angle DEB = \angle DFC$. Lines $DF,\ DE$ intersect $AB,\ AC$ at points $M,\ N$, respectively. Denote $(I_1),\ (I_2)$ by the circumcircles of triangles $DEM,\ DFN$ in that order. The circle (J_1) touches (I_1) internally at D and touches AB at K, circle (J_2) touches (I_2) internally at D and touches AC at H. P is the intersection of $(I_1),\ (I_2)$ different from D. Q is the intersection of $(J_1),\ (J_2)$ different from D.
a. Prove that all points $D,\ P,\ Q$ lie on the same line.
b. The circumcircles of triangles $AEF,\ AHK$ intersect at $A,\ G$. (AEF) also cut AQ at $A,\ L$. Prove that the tangent at D of (DQG) cuts EF at a point on (DLG).

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3 An investor has two rectangular pieces of land of size 120×100.
a. On the first land, she want to build a house with a rectangular base of size 25×35 and nines circular flower pots with diameter 5 outside the house. Prove that even the flower pots positions are chosen arbitrary on the land, the remaining land is still sufficient to build the desired house.
b. On the second land, she want to construct a polygonal fish pond such that the distance from an arbitrary point on the land, outside the pond, to the nearest pond edge is not over 5. Prove that the perimeter of the pond is not smaller than $440 - 20\sqrt{2}$.

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4 On the Cartesian plane the curve (C) has equation $x^2 = y^3$. A line d varies on the plane such that d always cut (C) at three distinct points with x-coordinates $x_1,\ x_2,\ x_3$.
a. Prove that the following quantity is a constant:

$$\sqrt[3]{\frac{x_1 x_2}{x_3^2}} + \sqrt[3]{\frac{x_2 x_3}{x_1^2}} + \sqrt[3]{\frac{x_3 x_1}{x_2^2}}.$$

b. Prove the following inequality:

$$\sqrt[3]{\frac{x_1^2}{x_2x_3}}+\sqrt[3]{\frac{x_2^2}{x_3x_1}}+\sqrt[3]{\frac{x_3^2}{x_3x_1}}<-\frac{15}{4}.$$

Second day 12th January 2018

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5 For two positive integers n and d, let $S_n(d)$ be the set of all ordered d-tuples $(x_1, x_2, \dots, x_d)$ that satisfy all of the following conditions:
i. $x_i \in \{1, 2, \dots, n\}$ for every $i \in \{1, 2, \dots, d\}$;
ii. $x_i \neq x_{i+1}$ for every $i \in \{1, 2, \dots, d-1\}$;
iii. There does not exist $i, j, k, l \in \{1, 2, \dots, d\}$ such that $i < j < k < l$ and $x_i = x_k$, $x_j = x_l$;
a. Compute $|S_3(5)|$
b. Prove that $|S_n(d)| > 0$ if and only if $d \leq 2n-1$.

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6 The sequence (x_n) is defined as follows:

$$x_0 = 2,\ x_1 = 1,\ x_{n+2} = x_{n+1} + x_n$$

for every non-negative integer n.
a. For each $n \geq 1$, prove that x_n is a prime number only if n is a prime number or n has no odd prime divisors
b. Find all non-negative pairs of integers (m, n) such that $x_m | x_n$.

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7 Acute scalene triangle ABC has G as its centroid and O as its circumcenter. Let H_a, H_b, H_c be the projections of A, B, C on respective opposite sides and D, E, F be the midpoints of BC, CA, AB in that order. $\overrightarrow{GH_a}$, $\overrightarrow{GH_b}$, $\overrightarrow{GH_c}$ intersect (O) at X, Y, Z respectively.
a. Prove that the circle (XCE) pass through the midpoint of BH_a
b. Let M, N, P be the midpoints of AX, BY, CZ respectively. Prove that $\overleftrightarrow{DM}$, $\overleftrightarrow{EN}$, $\overleftrightarrow{FP}$ are concurrent.
