The problems from the CCA Math Bonanza held on 1/20/2018

www.artofproblemsolving.com/community/c596121

by trumpeter

–	Individual Round
I1	What is the tens digit of the sum $$(1!)^2+(2!)^2+(3!)^2+\ldots+(2018!)^2?$$ *2018 CCA Math Bonanza Individual Round#1*
I2	Let P be the product of the first 50 nonzero square numbers. Find the largest integer k such that 7^k divides P. *2018 CCA Math Bonanza Individual Round#2*
I3	A Louis Vuitton store in Shanghai had a number of pairs of sunglasses which cost an average of \$$900$ per pair. LiAngelo Ball stole a pair which cost \$$2000$. Afterwards, the average cost of sunglasses in the store dropped to \$$890$ per pair. How many pairs of sunglasses were in the store before LiAngelo Ball stole? *2018 CCA Math Bonanza Individual Round#3*
I4	Zadam Heng bets Saniel Dun that he can win in a free throw contest. Zadam shoots until he has made 5 shots. He wins if this takes him 9 or fewer attempts. The probability that Zadam makes any given attempt is $\frac{1}{2}$. What is the probability that Zadam Heng wins the bet? *2018 CCA Math Bonanza Individual Round#4*
I5	Determine all positive numbers x such that $$\frac{16}{x+2}+\frac{4}{x+0}+\frac{9}{x+1}+\frac{100}{x+8}=19.$$ *2018 CCA Math Bonanza Individual Round#5*
I6	A lumberjack is building a non-degenerate triangle out of logs. Two sides of the triangle have lengths $\log 101$ and $\log 2018$. The last side of his triangle has side length $\log n$, where n is an integer. How many possible values are there for n? *2018 CCA Math Bonanza Individual Round#6*

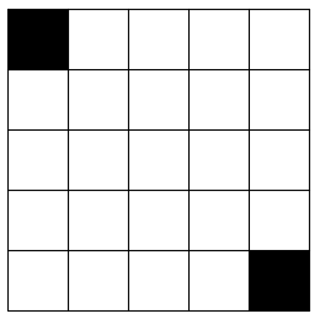
I7 Find all values of a such that the two polynomials

$$x^2 + ax - 1 \qquad \text{and} \qquad x^2 - x + a$$

share at least 1 root.

2018 CCA Math Bonanza Individual Round#7

I8 The New York Times Mini Crossword is a 5×5 grid with the top left and bottom right corners shaded. Each row and column has a clue given (so that there are 10 clues total). Jeffrey has a $\frac{1}{2}$ chance of knowing the answer to each clue. What is the probability that he can fill in every unshaded square in the crossword?



2018 CCA Math Bonanza Individual Round#8

I9 What is the area of the smallest possible square that can be drawn around a regular hexagon of side length 2 such that the hexagon is contained entirely within the square?

2018 CCA Math Bonanza Individual Round#9

I10 In the land of Chaina, people pay each other in the form of links from chains. Fiona, originating from Chaina, has an open chain with 2018 links. In order to pay for things, she decides to break up the chain by choosing a number of links and cutting them out one by one, each time creating 2 or 3 new chains. For example, if she cuts the 1111th link out of her chain first, then she will have 3 chains, of lengths 1110, 1, and 907. What is the least number of links she needs to remove in order to be able to pay for anything costing from 1 to 2018 links using some combination of her chains?

2018 CCA Math Bonanza Individual Round#10

I11 Square $ABCD$ has side length 1; circle Γ is centered at A with radius 1. Let M be the midpoint of BC, and let N be the point on segment CD such that MN is tangent to Γ. Compute MN.

2018 CCA Math Bonanza Individual Round#11

I12 For how many integers $n \neq 1$ does $(n-1)^3$ divide $n^{2018(n-1)} - 1$?

2018 CCA Math Bonanza Individual Round#12

I13 $P(x)$ is a polynomial of degree at most 6 such that such that $P(1)$, $P(2)$, $P(3)$, $P(4)$, $P(5)$, $P(6)$, and $P(7)$ are 1, 2, 3, 4, 5, 6, and 7 in some order. What is the maximum possible value of $P(8)$?

2018 CCA Math Bonanza Individual Round#13

I14 Brian starts at the point $(1,0)$ in the plane. Every second, he performs one of two moves: he can move from (a,b) to $(a-b,a+b)$ or from (a,b) to $(2a-b,a+2b)$. How many different paths can he take to end up at $(28,-96)$?

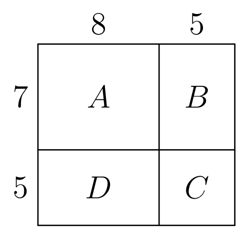
2018 CCA Math Bonanza Individual Round#14

I15 In a triangle ABC, let the B-excircle touch CA at E, C-excircle touch AB at F. If M is the midpoint of BC, then let the angle bisector of $\angle BAC$ meet BC, EF, ME, MF at D, P, E', F'. Suppose that the circumcircles of $\triangle EPE'$ and $\triangle FPF'$ meet again at a point Q and the circumcircle of $\triangle DPQ$ meets line EF again at X. If $BC = 10, CA = 20, AB = 18$, compute $|XE - XF|$.

2018 CCA Math Bonanza Individual Round#15

– Team Round

T1 In the diagram of rectangles below, with lengths as labeled, let A be the area of the rectangle labeled A, and so on. Find $36A + 6B + C + 6D$.



2018 CCA Math Bonanza Team Round#1

T2 Arnold has plates weighing 5, 15, 25, 35, or 45 pounds. He lifts a barbell, which consists of a 45-pound bar and any number of plates that he has. Vlad looks at Arnold's bar and is impressed to see him bench-press 600 pounds. Unfortunately, Vlad mistook each plate on Arnold's bar for the plate one size heavier, and Arnold was actually lifting 470 pounds. How many plates did Arnold have on the bar?

2018 CCA Math Bonanza Team Round#2

T3 In the game of Avalon, there are 10 players, 4 of which are bad. A *quest* is a subset of those players. A quest fails if it contains at least one bad player. A randomly chosen quest of 3 players happens to fail. What is the probability that there is exactly one bad player in the failed quest?

2018 CCA Math Bonanza Team Round#3

T4 $ABCD$ is a convex quadrilateral with $AB = 36$, $CD = 9$, $DA = 39$, and $BD = 15$. Given that $\angle C$ is right, compute the area of $ABCD$.

2018 CCA Math Bonanza Team Round#4

T5 Call a day a *perfect* day if the sum of the digits of the month plus sum of the digits of the day equals the sum of digits of the year. For example, February 28th, 2028 is a perfect day because $2+2+8 = 2+0+2+8$. Find the number of perfect days in 2018.

2018 CCA Math Bonanza Team Round#5

T6 Circle Γ with radius 1 is centered at point A on the circumference of circle ω with radius 7. Suppose that point P lies on ω with $AP = 4$. Determine the product of the distances from P to the two intersections of ω and Γ.

2018 CCA Math Bonanza Team Round#6

T7 Compute

$$\sum_{i=0}^{\frac{q-1}{2}} \left\lfloor \frac{ip}{q} \right\rfloor + \sum_{j=0}^{\frac{p-1}{2}} \left\lfloor \frac{jq}{p} \right\rfloor$$

if $p = 51$ and $q = 81$.

2018 CCA Math Bonanza Team Round#7

T8 A rectangular prism with positive integer side lengths formed by stacking unit cubes is called *bipartisan* if the same number of unit cubes can be seen on the surface as those which cannot be seen on the surface. How many non-congruent bipartisan rectangular prisms are there?

2018 CCA Math Bonanza Team Round#8

T9 21 Savage has a 12 car garage, with a row of spaces numbered $1, 2, 3, \dots, 12$. How many ways can he choose 6 of them to park his 6 identical cars in, if no 3 spaces with consecutive numbers may be all occupied?

2018 CCA Math Bonanza Team Round#9

T10 The irrational number $\alpha > 1$ satisfies $\alpha^2 - 3\alpha - 1 = 0$. Given that there is a fraction $\frac{m}{n}$ such that $n < 500$ and $\left|\alpha - \frac{m}{n}\right| < 3 \cdot 10^{-6}$, find m.

2018 CCA Math Bonanza Team Round#10

– Lightning Round

L1.1 Let $A=1,B=2,\dots,Z=26$. Compute $BONANZA$, where the result is the product of the numbers represented by each letter.

2018 CCA Math Bonanza Lightning Round#1.1

L1.2 The CCA Math Banana$^{\text{TM}}$ costs \$$100$. The cost rises 10

2018 CCA Math Bonanza Lightning Round#1.2

L1.3 $ABCDEF$ is a hexagon inscribed in a circle such that the measure of $\angle ACE$ is 90°. What is the average of the measures, in degrees, of $\angle ABC$ and $\angle CDE$?

2018 CCA Math Bonanza Lightning Round#1.3

L1.4 What is the sum of all distinct values of x that satisfy $x^4-x^3-7x^2+13x-6=0$?

2018 CCA Math Bonanza Lightning Round#1.4

L2.1 Let S be the set of the first 2018 positive integers, and let T be the set of all distinct numbers of the form ab, where a and b are distinct members of S. What is the 2018th smallest member of T?

2018 CCA Math Bonanza Lightning Round#2.1

L2.2 Points X,Y,Z lie on a line in this order and point P lies off this line such that $\angle XPY=\angle PZY$. If $XY=4$ and $YZ=5$, compute PX.

2018 CCA Math Bonanza Lightning Round#2.2

L2.3 On January $20, 2018$, Sally notices that her 7 children have ages which sum to a perfect square: their ages are 1, 3, 5, 7, 9, 11, and 13, with $1+3+5+7+9+11+13=49$. Let N be the age of the youngest child the next year the sum of the 7 children's ages is a perfect square on January 20th, and let P be that perfect square. Find $N+P$.

2018 CCA Math Bonanza Lightning Round#2.3

L2.4 Alex, Bertha, Cameron, Dylan, and Ellen each have a different toy. Each kid puts each of their own toys into a large bag. The toys are then randomly distributed such that each kid receives a toy. How many ways are there for exactly one kid to get the same toy that they put in?

2018 CCA Math Bonanza Lightning Round#2.4

L3.1 The number 16^4+16^2+1 is divisible by four distinct prime numbers. Compute the sum of these four primes.

2018 CCA Math Bonanza Lightning Round#3.1

L3.2 How many positive integers $n \leq 100$ satisfy $\lfloor n\pi \rfloor = \lfloor (n-1)\pi \rfloor + 3$? Here $\lfloor x \rfloor$ is the greatest integer less than or equal to x; for example, $\lfloor \pi \rfloor = 3$.

2018 CCA Math Bonanza Lightning Round#3.2

L3.3 On January 15 in the stormy town of Stormville, there is a 50% chance of rain. Every day, the probability of it raining has a 50% chance of being $\frac{2017}{2016}$ times that of the previous day (or 100% if this new quantity is over 100%) and a 50% chance of being $\frac{1007}{2016}$ times that of the previous day. What is the probability that it rains on January 20?

2018 CCA Math Bonanza Lightning Round#3.3

L3.4 Consider equilateral triangle ABC with side length 1. Suppose that a point P in the plane of the triangle satisfies

$$2AP = 3BP = 3CP = \kappa$$

for some constant κ. Compute the sum of all possible values of κ.

2018 CCA Math Bonanza Lightning Round#3.4

L4.1 Let S be the set of all ordered triples (a,b,c) of positive integers such that $(b-c)^2+(c-a)^2+(a-b)^2=2018$ and $a+b+c \leq M$ for some positive integer M. Given that $\sum_{(a,b,c)\in S} a = k$, what is

$$\sum_{(a,b,c)\in S} a\left(a^2-bc\right)$$

in terms of k?

2018 CCA Math Bonanza Lightning Round#4.1

L4.2 A subset of $\{1,2,3,\ldots,2017,2018\}$ has the property that none of its members are 5 times another. What is the maximum number of elements that such a subset could have?

2018 CCA Math Bonanza Lightning Round#4.2

L4.3 ABC is an isosceles triangle with $AB = AC$. Point D is constructed on AB such that $\angle BCD = 15^\circ$. Given that $BC = \sqrt{6}$ and $AD = 1$, find the length of CD.

2018 CCA Math Bonanza Lightning Round#4.3

L4.4 Alice and Billy are playing a game on a number line. They both start at 0. Each turn, Alice has a $\frac{1}{2}$ chance of moving 1 unit in the positive direction, and a $\frac{1}{2}$ chance of moving 1 unit in the negative direction, while Billy has a $\frac{2}{3}$ chance of moving 1 unit in the positive direction, and a $\frac{1}{3}$ chance of moving 1 unit in the negative direction. Alice and Billy alternate turns, with Alice going first. If a player reaches 2, they win and the game ends, but if they reach -2, they lose and the other player wins, and the game ends. What is the probability that Billy wins?

2018 CCA Math Bonanza Lightning Round#4.4

L5.1 Estimate the number of five-card combinations from a standard 52-card deck that contain a pair (two cards with the same number).

An estimate of E earns $2e^{-\frac{|A-E|}{20000}}$ points, where A is the actual answer.

2018 CCA Math Bonanza Lightning Round#5.1

L5.2 Two circles of equal radii are drawn to intersect at X and Y. Suppose that the two circles bisect each other's areas. If the measure of minor arc $\widehat{XY}$ is θ degrees, estimate $\lfloor 1000\theta \rfloor$.

An estimate of E earns $2e^{-\frac{|A-E|}{50000}}$ points, where A is the actual answer.

2018 CCA Math Bonanza Lightning Round#5.2

L5.3 Choose an integer n from 1 to 10 inclusive as your answer to this problem. Let m be the number of distinct values in $\{1, 2, \dots, 10\}$ chosen by all teams at the Math Bonanza for this problem which are greater than or equal to n. Your score on this problem will be $\frac{mn}{15}$. For example, if 5 teams choose 1, 2 teams choose 2, and 6 teams choose 3 with these being the only values chosen, and you choose 2, you will receive $\frac{4}{15}$ points.

2018 CCA Math Bonanza Lightning Round#5.3

L5.4 Welcome to the **USAYNO**, a twelve-part question where each part has a yes/no answer. If you provide C correct answers, your score on this problem will be $\frac{C}{6}$.

Your answer should be a twelve-character string containing 'Y' (for yes) and 'N' (for no). For instance if you think a, c, and f are 'yes' and the rest are 'no', you should answer YNYNNYNNNNNN.

(a) Is there a positive integer n such that the sum of the digits of $2018n + 1337$ in base 10 is 2018 more than the sum of the digits of $2018n + 1337$ in base 4?

(b) Is there a fixed constant θ such that for all triangles ABC with

$$2018AB^2 = 2018CA^2 + 2017CA \cdot CB + 2018CB^2,$$

one of the angles of ABC is θ?

(c) Adam lists out every possible way to arrange the letters of "CCACCACCA" (including the given arrangement) at 1 arrangement every 5 seconds. Madam lists out every possible way to arrange the letters of "CCACCAA" (including the given arrangement) at 1 arrangement every 12 seconds. Does Adam finish first?

(d) Do there exist real numbers a, b, c, none of which is the average of the other two, such that

$$\frac{1}{b+c-2a}+\frac{1}{c+a-2b}+\frac{1}{a+b-2c}=0?$$

(e) Let $f(x)=\frac{2^x-2}{x}-1$. Is there an integer n such that

$$f(n), f(f(n)), f(f(f(n))), \ldots$$

are all integers?

(f) In an elementary school with 2585 students and 159 classes (every student is in exactly one class), each student reports the size of their class. The principal of the school takes the average of all of these numbers and calls it X. The principal then computes the average size of each class and calls it Y. Is it necessarily true that $X>Y$?

(g) Six sticks of lengths 3, 5, 7, 11, 13, and 17 are put together to form a hexagon. From a point inside the hexagon, a circular water balloon begins to expand and will stop expanding once it hits any stick. Is it possible that once the balloon stops expanding, it is touching each of the six sticks?

(h) A coin is biased so that it flips heads and tails (and only heads or tails) each with a positive rational probability (not necessarily $\frac{1}{2}$). Is it possible that on average, it takes exactly twice as long to flip two heads in a row as it is to flip two tails in a row?

(i) Does there exist a base b such that 2018_b is prime?

(j) Does there exist a sequence of 2018 distinct real numbers such that no 45 terms (not necessarily consecutive) can be examined, in order, and be in strictly increasing or strictly decreasing order?

(k) Does there exist a scalene triangle ABC such that there exist two distinct rectangles $PQRS$ inscribed in $\triangle ABC$ with $P\in AB$, $Q, R\in BC$, $S\in AC$ such that the angle bisectors of $\angle PAS$, $\angle PQR$, and $\angle SRQ$ concur?

(l) For three vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ with $\mathbf{u}_i=(x_{i,1}, x_{i,2}, x_{i,3}, x_{i,4})$, define

$$f(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)=1-\prod_{j=1}^{4}\left(1+(x_{2,j}-x_{3,j})^2+(x_{3,j}-x_{1,j})^2+(x_{1,j}-x_{2,j})^2\right).$$

Are there any sequences $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{18}$ of distinct vectors with four components, with all components in $\{1,2,3\}$, such that

$$\prod_{1\le i<j<k\le 18} f(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_k)\equiv 1 \pmod 3?$$

2018 CCA Math Bonanza Lightning Round#5.4

– Tiebreaker Round

TB1 What is the maximum number of diagonals of a regular 12-gon which can be selected such that no two of the chosen diagonals are perpendicular?

Note: sides are not diagonals and diagonals which intersect outside the 12-gon at right angles are still considered perpendicular.

2018 CCA Math Bonanza Tiebreaker Round#1

TB2 Define a sequence of polynomials $P_0\left(x\right)=x$ and $P_k\left(x\right)=P_{k-1}\left(x\right)^2-\left(-1\right)^kk$ for each $k\geq1$. Also define $Q_0\left(x\right)=x$ and $Q_k\left(x\right)=Q_{k-1}\left(x\right)^2+\left(-1\right)^kk$ for each $k\geq1$. Compute the product of the distinct real roots of

$$P_1\left(x\right)Q_1\left(x\right)P_2\left(x\right)Q_2\left(x\right)\cdots P_{2018}\left(x\right)Q_{2018}\left(x\right).$$

2018 CCA Math Bonanza Tiebreaker Round#2

TB3 Given that 5^{2018} has 1411 digits and starts with 3 (the leftmost non-zero digit is 3), for how many integers $1\leq n\leq2017$ does 5^n start with 1?

2018 CCA Math Bonanza Tiebreaker Round#3

TB4 Triangle ABC is a triangle with side lengths 13, 14, and 15. A point Q is chosen uniformly at random in the interior of $\triangle{ABC}$. Choose a random ray (of uniformly random direction) with endpoint Q and let it intersect the perimeter of $\triangle{ABC}$ at P. What is the expected value of QP^2?

2018 CCA Math Bonanza Tiebreaker Round#4