India National Olympiad 2018

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by integrated_JRC

1 Let ABC be a non-equilateral triangle with integer sides. Let D and E be respectively the mid-points of BC and CA ; let G be the centroid of ΔABC. Suppose, D, C, E, G are concyclic. Find the least possible perimeter of ΔABC.

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2 For any natural number n, consider a $1 \times n$ rectangular board made up of n unit squares. This is covered by 3 types of tiles : 1×1 red tile, 1×1 green tile and 1×2 domino. (For example, we can have 5 types of tiling when $n = 2$: red-red ; red-green ; green-red ; green-green ; and blue.) Let t_n denote the number of ways of covering $1 \times n$ rectangular board by these 3 types of tiles. Prove that, t_n divides t_{2n+1}.

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3 Let Γ_1 and Γ_2 be two circles with respective centres O_1 and O_2 intersecting in two distinct points A and B such that $\angle O_1AO_2$ is an obtuse angle. Let the circumcircle of ΔO_1AO_2 intersect Γ_1 and Γ_2 respectively in points $C(\neq A)$ and $D(\neq A)$. Let the line CB intersect Γ_2 in E ; let the line DB intersect Γ_1 in F. Prove that, the points C, D, E, F are concyclic.

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4 Find all polynomials with real coefficients $P(x)$ such that $P(x^2 + x + 1)$ divides $P(x^3 - 1)$.

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5 There are $n \geq 3$ girls in a class sitting around a circular table, each having some apples with her. Every time the teacher notices a girl having more apples than both of her neighbours combined, the teacher takes away one apple from that girl and gives one apple each to her neighbours. Prove that, this process stops after a finite number of steps.
(Assume that, the teacher has an abundant supply of apples.)

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6 Let $\mathbb{N}$ denote set of all natural numbers and let $f : \mathbb{N} \to \mathbb{N}$ be a function such that

(a)$f(mn) = f(m).f(n)$ for all $m, n \in \mathbb{N}$;

(b)$m + n$ divides $f(m) + f(n)$ for all $m, n \in \mathbb{N}$.

Prove that, there exists an odd natural number k such that $f(n) = n^k$ for all n in $\mathbb{N}$.
