www.artofproblemsolving.com/community/c602484
by Iaegolas, PlatinumFalcon, Headhunter, Scientist, basemfouda2002, Spotsjoy

– Paper 1

1 Hamilton Avenue has eight houses. On one side of the street are the houses numbered 1,3,5,7 and directly opposite are houses 2,4,6,8 respectively. An eccentric postman starts deliveries at house 1 and delivers letters to each of the houses, finally returning to house 1 for a cup of tea. Throughout the entire journey he must observe the following rules. The numbers of the houses delivered to must follow an odd-even-odd-even pattern throughout, each house except house 1 is visited exactly once (house 1 is visited twice) and the postman at no time is allowed to cross the road to the house directly opposite. How many different delivery sequences are possible?

2 Let $ABCD$ be a square. The line segment AB is divided internally at H so that $|AB| \cdot |BH| = |AH|^2$. Let E be the midpoints of AD and X be the midpoint of AH. Let Y be a point on EB such that XY is perpendicular to BE. Prove that $|XY| = |XH|$.

3 Find all positive integers n for which $n^8 + n + 1$ is a prime number.

4 Given an n-tuple of numbers $(x_1, x_2, \ldots, x_n)$ where each $x_i = +1$ or -1, form a new n-tuple

$$(x_1x_2, x_2x_3, x_3x_4, \ldots, x_nx_1),$$

and continue to repeat this operation. Show that if $n = 2^k$ for some integer $k \geq 1$, then after a certain number of repetitions of the operation, we obtain the n-tuple

$$(1, 1, 1, \ldots, 1).$$

5 Hello.

Suppose a, b, c are real numbers such that $a + b + c = 0$ and $a^2 + b^2 + c^2 = 1$.
Prove that $a^2b^2c^2 \leq \frac{1}{54}$ and determine the cases of equality.

– Paper 2

1 Let $P(x)$ be a polynomial with rational coefficients. Prove that there exists a positive integer n such that the polynomial $Q(x)$ defined by

$$Q(x) = P(x + n) - P(x)$$

has integer coefficients.

---

2 For any positive integer n define

$$E(n) = n(n+1)(2n+1)(3n+1)\cdots(10n+1).$$

Find the greatest common divisor of $E(1), E(2), E(3), \dots, E(2009)$.

---

3 Find all pairs (a, b) of positive integers such that $(ab)^2 - 4(a+b)$ is the square of an integer.

---

4 At a strange party, each person knew exactly 22 others.
For any pair of people X and Y who knew each other, there was no other person at the party that they both knew.
For any pair of people X and Y who did not know one another, there were exactly 6 other people that they both knew.
How many people were at the party?

---

5 In the triangle ABC we have $|AB| < |AC|$. The bisectors of the angles at B and C meet AC and AB at D and E respectively. BD and CE intersect at the incenter I of $\triangle ABC$.

Prove that $\angle BAC = 60^\circ$ if and only if $|IE| = |ID|$
