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– Paper 1

1 Find the least k for which the number 2010 can be expressed as the sum of the squares of k integers.

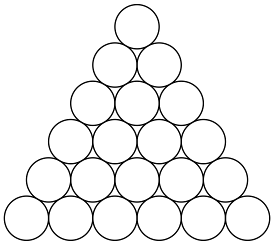
2 Let ABC be a triangle and let P denote the midpoint of the side BC. Suppose that there exist two points M and N interior to the side AB and AC respectively, such that

$$|AD| = |DM| = 2|DN|,$$

where D is the intersection point of the lines MN and AP. Show that $|AC| = |BC|$.

3 Suppose x, y, z are positive numbers such that $x + y + z = 1$. Prove that
(a) $xy + yz + xz \geq 9xyz$;
(b) $xy + yz + xz < \frac{1}{4} + 3xyz$;

4 The country of Harpland has three types of coins: green, white and orange. The unit of currency in Harpland is the shilling. Any coin is worth a positive integer number of shillings, but coins of the same color may be worth different amounts. A set of coins is stacked in the form of an equilateral triangle of side n coins, as shown below for the case of $n = 6$.



The stacking has the following properties:

(a) no coin touches another coin of the same color;

(b) the total worth, in shillings, of the coins lying on any line parallel to one of the sides of the triangle is divisible by by three.

Prove that the total worth in shillings of the *green* coins in the triangle is divisible by three.

5 Find all polynomials $f(x) = x^3+bx^2+cx+d$, where $b, c, d,$ are real numbers, such that $f(x^2-2) = -f(-x)f(x)$.

– Paper 2

1 There are 14 boys in a class. Each boy is asked how many other boys in the class have his first name, and how many have his last name. It turns out that each number from 0 to 6 occurs among the answers.

Prove that there are two boys in the class with the same first name and the same last name.

2 For each odd integer $p \geq 3$ find the number of real roots of the polynomial

$$f_p(x) = (x-1)(x-2)\cdots(x-p+1)+1.$$

3 In triangle ABC we have $|AB| = 1$ and $\angle ABC = 120^\circ$. The perpendicular line to AB at B meets AC at D such that $|DC| = 1$. Find the length of AD.

4 Let $n \geq 3$ be an integer and $a_1, a_2, \dots, a_n$ be a finite sequence of positive integers, such that, for $k = 2, 3, \dots, n$

$$n(a_k + 1) - (n-1)a_{k-1} = 1.$$

Prove that a_n is not divisible by $(n-1)^2$.

5 Suppose a, b, c are the side lengths of a triangle ABC. Show that

$$x = \sqrt{a(b+c-a)}, y = \sqrt{b(c+a-b)}, z = \sqrt{c(a+b-c)}$$

are the side lengths of an acute-angled triangle XYZ, with the same area as ABC, but with a smaller perimeter, unless ABC is equilateral.