National Olympiad Second Round 2017

www.artofproblemsolving.com/community/c602910

by Eray, KereMath

Day 1 December 16th

1 A wedding is going to be held in a city with 25 types of meals, to which some of the 2017 citizens will be invited. All of the citizens like some meals and each meal is liked by at least one person. A "$suitable\ list$" is a set of citizens, such that each meal is liked by at least one person in the set. A "$kamber\ group$" is a set that contains at least one person from each "$suitable\ list$". Given a "$kamber\ group$", which has no subset (other than itself) that is also a "$kamber\ group$", prove that there exists a meal, which is liked by everyone in the group.

2 Let $ABCD$ be a quadrilateral such that line AB intersects CD at X. Denote circles with inradius r_1 and centers A, B as w_a and w_b with inradius r_2 and centers C, D as w_c and w_d. w_a intersects w_d at P, Q. w_b intersects w_c at R, S. Prove that if $XA.XB + r_2^2 = XC.XD + r_1^2$, then P, Q, R, S are cyclic.

3 Denote the sequence $a_{i,j}$ in positive reals such that $a_{i,j}.a_{j,i} = 1$. Let $c_i = \sum_{k=1}^{n} a_{k,i}$. Prove that $1 \geq \sum_{i=1}^{n} \frac{1}{c_i}$

Day 2 December 17th

4 Let $d(n)$ be number of prime divisors of n. Prove that one can find k, m positive integers for any positive integer n such that $k - m = n$ and $d(k) - d(m) = 1$

5 Let $x_0, \ldots, x_{2017}$ are positive integers and $x_{2017} \geq \cdots \geq x_0 = 1$ such that $A = \{x_1, \ldots, x_{2017}\}$ consists of exactly 25 different numbers. Prove that $\sum_{i=2}^{2017} (x_i - x_{i-2}) x_i \geq 623$, and find the number of sequences that holds the case of equality.

6 Finite number of 2017 units long sticks are fixed on a plate. Each stick has a bead that can slide up and down on it. Beads can only stand on integer heights $(1, 2, 3, ..., 2017)$. Some of the bead pairs are connected with elastic bands. $The\ young\ ant$ can go to every bead, starting from any bead by using the elastic bands. $The\ old\ ant$ can use an elastic band if the difference in height of the beads which are connected by the band, is smaller than or equal to 1. If the heights of the beads which are connected to each other are different, we call it $valid\ situation$. If there exists at least one $valid\ situation$, prove that we can create a $valid\ situation$, by arranging the heights of the beads, in which $the\ old\ ant$ can go to every bead, starting from any bead.