Sharygin Geometry Olympiad 2017

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Grade 8

Day 1

1 Let $ABCD$ be a cyclic quadrilateral with $AB = BC$ and $AD = CD$. A point M lies on the minor arc CD of its circumcircle. The lines BM and CD meet at point P, the lines AM and BD meet at point Q. Prove that $PQ \parallel AC$.

2 Let H and O be the orthocenter and circumcenter of an acute-angled triangle ABC, respectively. The perpendicular bisector of BH meets AB and BC at points A_1 and C_1, respectively. Prove that OB bisects the angle A_1OC_1.

3 Let AD, BE and CF be the medians of triangle ABC. The points X and Y are the reflections of F about AD and BE, respectively. Prove that the circumcircles of triangles BEX and ADY are concentric.

4 Alex dissects a paper triangle into two triangles. Each minute after this he dissects one of obtained triangles into two triangles. After some time (at least one hour) it appeared that all obtained triangles were congruent. Find all initial triangles for which this is possible.

Day 2

5 A square $ABCD$ is given. Two circles are inscribed into angles A and B, and the sum of their diameters is equal to the sidelength of the square. Prove that one of their common tangents passes through the midpoint of AB.

6 A median of an acute-angled triangle dissects it into two triangles. Prove that each of them can be covered by a semidisc congruent to a half of the circumdisc of the initial triangle.

7 Let $A_1A_2 \dots A_{13}$ and $B_1B_2 \dots B_{13}$ be two regular 13-gons in the plane such that the points B_1 and A_{13} coincide and lie on the segment A_1B_{13}, and both polygons lie in the same semiplane with respect to this segment. Prove that the lines $A_1A_9, B_{13}B_8$ and A_8B_9 are concurrent.

8 Let $ABCD$ be a square, and let P be a point on the minor arc CD of its circumcircle. The lines PA, PB meet the diagonals BD, AC at points K, L respectively. The points M, N are the projections of K, L respectively to CD, and Q is the common point of lines KN and ML. Prove that PQ bisects the segment AB.

Grade 9

Day 1

1 Let ABC be a regular triangle. The line passing through the midpoint of AB and parallel to AC meets the minor arc AB of the circumcircle at point K. Prove that the ratio $AK : BK$ is equal to the ratio of the side and the diagonal of a regular pentagon.

2 Let I be the incenter of a triangle ABC, M be the midpoint of AC, and W be the midpoint of arc AB of the circumcircle not containing C. It is known that $\angle AIM = 90^\circ$. Find the ratio $CI : IW$.

3 The angles B and C of an acute-angled triangle ABC are greater than 60°. Points P, Q are chosen on the sides AB, AC respectively so that the points A, P, Q are concyclic with the orthocenter H of the triangle ABC. Point K is the midpoint of PQ. Prove that $\angle BKC > 90^\circ$.

Proposed by A. Mudgal

4 Points M and K are chosen on lateral sides AB, AC of an isosceles triangle ABC and point D is chosen on BC such that $AMDK$ is a parallelogram. Let the lines MK and BC meet at point L, and let X, Y be the intersection points of AB, AC with the perpendicular line from D to BC. Prove that the circle with center L and radius LD and the circumcircle of triangle AXY are tangent.

– Day 2

5 Let BH_b, CH_c be altitudes of an acute-angled triangle ABC. The line H_bH_c meets the circumcircle of ABC at points X and Y. Points P, Q are the reflections of X, Y about AB, AC respectively. Prove that $PQ \parallel BC$.

Proposed by Pavel Kozhevnikov

6 Let ABC be a right-angled triangle ($\angle C = 90^\circ$) and D be the midpoint of an altitude from C. The reflections of the line AB about AD and BD, respectively, meet at point F. Find the ratio $S_{ABF} : S_{ABC}$.
Note: S_α means the area of α.

7 Let a and b be parallel lines with 50 distinct points marked on a and 50 distinct points marked on b. Find the greatest possible number of acute-angled triangles all of whose vertices are marked.

8 Let AK and BL be the altitudes of an acute-angled triangle ABC, and let ω be the excircle of ABC touching side AB. The common internal tangents to circles CKL and ω meet AB at points P and Q. Prove that $AP = BQ$.

Proposed by I.Frolov

Grade 10

Day 1

1 If two circles intersect at A, B and common tangents of them intesrsect circles at C, Dif $O_a$is circumcentre of ACD and O_b is circumcentre of BCD prove AB intersects O_aO_b at its midpoint

2 If ABC is acute triangle, prove distance from each vertex to corresponding excentre is less than sum of two greatest side of triangle

3 $ABCD$ is convex quadrilateral. If W_a is product of power of A about circle BCD and area of triangle BCD. And define W_b, W_c, W_d similarly.prove $W_a + W_b + W_c + W_d = 0$

4 Given triangle ABC and its incircle ω prove you can use just a ruler and drawing at most 8 lines to construct pointsA', B', C' on ω such that A, B', C' and B, C', A' and C, A', B' are collinear.

Day 2

5 10.5 Let BB', CC' be the altitudes of an acute triangle ABC. Two circles through A and C' are tangent to BC at points P and Q. Prove that A, B', P, Q are concyclic.

6 10.6 Let the insphere of a pyramid $SABC$ touch the faces SAB, SBC, SCA at D, E, F respectively. Find all the possible values of the sum of the angles SDA, SEB, SFC.

7 10.7 A quadrilateral $ABCD$ is circumscribed around the circle ω centered at I and inscribed into the circle Γ. The lines AB, CD meet at point P, and the lines BC, AD meet at point Q. Prove that the circles $\odot(PIQ)$ and Γ are orthogonal.

8 10.8 Suppose S is a set of points in the plane, $|S|$ is even; no three points of S are collinear. Prove that S can be partitioned into two sets S_1 and S_2 so that their convex hulls have equal number of vertices.

– First Round

– Grade 8

P1 Mark on a cellular paper four nodes forming a convex quadrilateral with the sidelengths equal to four different primes.
(Proposed by A.Zaslavsky)

P2 A circle cuts off four right-angled triangles from rectangle $ABCD$.Let A_0, B_0, C_0 and D_0 be the midpoints of the correspondent hypotenuses. Prove that $A_0C_0 = B_0D_0$
Proosed by L.Shteingarts

P3 Let I be the incenter of triangle ABC; H_B, H_C the orthocenters of triangles ACI and ABI respectively; K the touching point of the incircle with the side BC. Prove that H_B, H_C and K are collinear.
Proposed by M.Plotnikov

P4 A triangle ABC is given. Let C' be the vertex of an isosceles triangle ABC' with $\angle C' = 120^\circ$ constructed on the other side of AB than C, and B' be the vertex of an equilateral triangle ACB' constructed on the same side of AC as ABC. Let K be the midpoint of BB'
Find the angles of triangle KCC'.
Proposed by A.Zaslavsky

– Grades 89

P5 A segment AB is fixed on the plane. Consider all acute-angled triangles with side AB. Find the locus of
) the vertices of their greatest angles,
b) their incenters.

P6 Let $ABCD$ be a convex quadrilateral with $AC = BD = AD$; E and F the midpoints of AB and CD respectively; O the common point of the diagonals.Prove that EF passes through the touching points of the incircle of triangle AOD with AO and OD
Proposed by N.Moskvitin

P7 The circumcenter of a triangle lies on its incircle. Prove that the ratio of its greatest and smallest sides is less than two.
Proposed by B.Frenkin

P8 Let AD be the base of trapezoid $ABCD$. It is known that the circumcenter of triangle ABC lies on BD. Prove that the circumcenter of triangle ABD lies on AC.
Proposed by Ye.Bakayev

P9 Let C_0 be the midpoint of hypotenuse AB of triangle ABC; AA_1, BB_1 the bisectors of this triangle; I its incenter. Prove that the lines C_0I and A_1B_1 meet on the altitude from C.
Proposed by A.Zaslavsky

– Grades 810

P10 Points K and L on the sides AB and BC of parallelogram $ABCD$ are such that $\angle AKD = \angle CLD$. Prove that the circumcenter of triangle BKL is equidistant from A and C.

Proposed by I.I.Bogdanov

– Grades 811

P11 A finite number of points is marked on the plane. Each three of them are not collinear. A circle is circumscribed around each triangle with marked vertices. Is it possible that all centers of these circles are also marked?
Proposed by A.Tolesnikov

– Grades 910

P12 Let AA_1, CC_1 be the altitudes of triangle ABC, B_0 the common point of the altitude from B and the circumcircle of ABC; and Q the common point of the circumcircles of ABC and $A_1C_1B_0$, distinct from B_0. Prove that BQ is the symmedian of ABC.
Proposed by D.Shvetsov

– Grades 911

P13 Two circles pass through points A and B. A third circle touches both these circles and meets AB at points C and D. Prove that the tangents to this circle at these points are parallel to the common tangents of two given circles.
Proposed by A.Zaslavsky

P14 Let points B and C lie on the circle with diameter AD and center O on the same side of AD. The circumcircles of triangles ABO and CDO meet BC at points F and E respectively. Prove that $R^2 = AF.DE$, where R is the radius of the given circle.
Proposed by N.Moskvitin

P15 Let ABC be an acute-angled triangle with incircle ω and incenter I. Let ω touch AB, BC and CA at points D, E, F respectively. The circles ω_1 and ω_2 centered at J_1 and J_2 respectively are inscribed into ADIF and $BDIE$. Let J_1J_2 intersect AB at point M. Prove that CD is perpendicular to IM.

P16 The tangents to the circumcircle of triangle ABC at A and B meet at point D. The circle passing through the projections of D to BC, CA, AB, meet AB for the second time at point C'. Points A', B' are defined similarly. Prove that AA', BB', CC' concur.

P17 Using a compass and a ruler, construct a point K inside an acute-angled triangle ABC so that $\angle KBA = 2\angle KAB$ and $\angle KBC = 2\angle KCB$.

P18 Let L be the common point of the symmedians of triangle ABC, and BH be its altitude. It is known that $\angle ALH = 180^o - 2\angle A$. Prove that $\angle CLH = 180^o - 2\angle C$.

– Grades 1011

P19 Let cevians AA', BB' and CC' of triangle ABC concur at point P. The circumcircle of triangle $PA'B'$ meets AC and BC at points M and N respectively, and the circumcircles of triangles $PC'B'$ and $PA'C'$ meet AC and BC for the second time respectively at points K and L. The line c passes through the midpoints of segments MN and KL. The lines a and b are defined similarly. Prove that a, b and c concur.

P20 Given a right-angled triangle ABC and two perpendicular lines x and y passing through the vertex A of its right angle. For an arbitrary point X on x define y_B and y_C as the reflections of y about XB and XC respectively. Let Y be the common point of y_b and y_c. Find the locus of Y (when y_b and y_c do not coincide).

P21 A convex hexagon is circumscribed about a circle of radius 1. Consider the three segments joining the midpoints of its opposite sides. Find the greatest real number r such that the length of at least one segment is at least r.

P22 Let P be an arbitrary point on the diagonal AC of cyclic quadrilateral $ABCD$, and PK, PL, PM, PN, PO be the perpendiculars from P to AB, BC, CD, DA, BD respectively. Prove that the distance from P to KN is equal to the distance from O to ML.

P23 Let a line m touch the incircle of triangle ABC. The lines passing through the incenter I and perpendicular to AI, BI, CI meet m at points A', B', C' respectively. Prove that AA', BB' and CC' concur.

– Grade 11

P24 Two tetrahedrons are given. Each two faces of the same tetrahedron are not similar, but each face of the first tetrahedron is similar to some face of the second one. Does this yield that these tetrahedrons are similar?