Japan MO Finals 2018

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1 Positive integers between 1 to 100 inclusive are written on a blackboard, each exactly once. One operation involves choosing two numbers a and b on the blackboard and erasing them, then writing the greatest common divisor of a^2+b^2+2 and a^2b^2+3. After a number of operations, there is only one positive integer left on the blackboard. Prove this number cannot be a perfect square.

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2 Given a scalene triangle $\triangle ABC$, D, E lie on segments AB, AC respectively such that $CA = CD, BA = BE$. Let ω be the circumcircle of $\triangle ADE$. P is the reflection of A across BC, and PD, PE meets ω again at X, Y respectively. Prove that BX and CY intersect on ω.

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3 Let $S = \{1, 2, \ldots, 999\}$. Consider a function $f : S \to S$, such that for any $n \in S$,

$$f^{n+f(n)+1}(n) = f^{nf(n)}(n) = n.$$

Prove that there exists $a \in S$, such that $f(a) = a$. Here $f^k(n) = \underbrace{f(f(\ldots f}_{k}(n)\ldots))$.

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4 Let n be an odd positive integer, and consider an infinite square grid. Prove that it is impossible to fill in one of $1, 2$ or 3 in every cell, which simultaneously satisfies the following conditions:
(1) Any two cells which share a common side does not have the same number filled in them.
(2) For any 1×3 or 3×1 subgrid, the numbers filled does not contain $1, 2, 3$ in that order be it reading from top to bottom, bottom to top, or left to right, or right to left.
(3) The sum of numbers of any $n\times n$ subgrid is the same.

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5 Let T be a positive integer. Find all functions $f : \mathbb{Z}^+\times\mathbb{Z}^+ \to \mathbb{Z}^+$, such that there exists integers $C_0, C_1, \ldots, C_T$ satisfying:
(1) For any positive integer n, the number of positive integer pairs (k, l) such that $f(k, l) = n$ is exactly n.
(2) For any $t = 0, 1, \ldots, T$, as well as for any positive integer pair (k, l), the equality $f(k+t, l+T-t) - f(k, l) = C_t$ holds.
