USAJMO 2018

www.artofproblemsolving.com/community/c615274

by green_dog_7983, Th3Numb3rThr33, hwl0304, CantonMathGuy, rrusczyk

–	**Day 1** April 18th	

1 For each positive integer n, find the number of n-digit positive integers that satisfy both of the following conditions:

- no two consecutive digits are equal, and
- the last digit is a prime.

2 Let a, b, c be positive real numbers such that $a+b+c=4\sqrt[3]{abc}$. Prove that

$$2(ab+bc+ca)+4\min(a^2,b^2,c^2)\geq a^2+b^2+c^2.$$

3 Let $ABCD$ be a quadrilateral inscribed in circle ω with $\overline{AC}\perp\overline{BD}$. Let E and F be the reflections of D over lines BA and BC, respectively, and let P be the intersection of lines BD and EF. Suppose that the circumcircle of $\triangle EPD$ meets ω at D and Q, and the circumcircle of $\triangle FPD$ meets ω at D and R. Show that $EQ=FR$.

– **Day 2** April 19th

4 Triangle ABC is inscribed in a circle of radius 2 with $\angle ABC\geq 90^\circ$, and x is a real number satisfying the equation $x^4+ax^3+bx^2+cx+1=0$, where $a=BC$, $b=CA$, $c=AB$. Find all possible values of x.

5 Let p be a prime, and let $a_1,\dots,a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1+k, a_2+2k, \dots, a_p+pk$$

produce at least $\tfrac{1}{2}p$ distinct remainders upon division by p.

Proposed by Ankan Bhattacharya

6 Karl starts with n cards labeled $1,2,3,\dots,n$ lined up in a random order on his desk. He calls a pair (a,b) of these cards *swapped* if $a>b$ and the card labeled a is to the left of the card labeled b. For instance, in the sequence of cards $3,1,4,2$, there are three swapped pairs of cards, $(3,1)$, $(3,2)$, and $(4,2)$.

He picks up the card labeled 1 and inserts it back into the sequence in the opposite position: if the card labeled 1 had i card to its left, then it now has i cards to its right. He then picks up the card labeled 2 and reinserts it in the same manner, and so on until he has picked up and put back each of the cards $1, 2, \dots, n$ exactly once in that order. (For example, the process starting at $3, 1, 4, 2$ would be $3, 1, 4, 2 \to 3, 4, 1, 2 \to 2, 3, 4, 1 \to 2, 4, 3, 1 \to 2, 3, 4, 1$.)

Show that, no matter what lineup of cards Karl started with, his final lineup has the same number of swapped pairs as the starting lineup.