10th RMM 2018

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–	Day 1
1	Let $ABCD$ be a cyclic quadrilateral an let P be a point on the side AB. The diagonals AC meets the segments DP at Q. The line through P parallel to CD mmets the extension of the side CB beyond B at K. The line through Q parallel to BD meets the extension of the side CB beyond B at L. Prove that the circumcircles of the triangles BKP and CLQ are tangent .
2	Determine whether there exist non-constant polynomials $P(x)$ and $Q(x)$ with real coefficients satisfying $$P(x)^{10}+P(x)^9=Q(x)^{21}+Q(x)^{20}.$$
3	Ann and Bob play a game on the edges of an infinite square grid, playing in turns. Ann plays the first move. A move consists of orienting any edge that has not yet been given an orientation. Bob wins if at any point a cycle has been created. Does Bob have a winning strategy?
–	Day 2
4	Let a,b,c,d be positive integers such that $ad \neq bc$ and $gcd(a,b,c,d)=1$. Let S be the set of values attained by $\gcd(an+b,cn+d)$ as n runs through the positive integers. Show that S is the set of all positive divisors of some positive integer.
5	Let n be positive integer and fix $2n$ distinct points on a circle. Determine the number of ways to connect the points with n arrows (oriented line segments) such that all of the following conditions hold: -each of the $2n$ points is a startpoint or endpoint of an arrow; -no two arrows intersect; and -there are no two arrows $\overrightarrow{AB}$ and $\overrightarrow{CD}$ such that A, B, C and D appear in clockwise order around the circle (not necessarily consecutively).
6	Fix a circle Γ, a line ℓ to tangent Γ, and another circle Ω disjoint from ℓ such that Γ and Ω lie on opposite sides of ℓ. The tangents to Γ from a variable point X on Ω meet ℓ at Y and Z. Prove that, as X varies over Ω, the circumcircle of XYZ is tangent to two fixed circles.