Greece National Olympiad 2018

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by Borbas

1 Let $(x_n), n \in \mathbb{N}$ be a sequence such that $x_{n+1} = 3x_n^3 + x_n, \forall n \in \mathbb{N}$ and $x_1 = \frac{a}{b}$ where a, b are positive integers such that $3 \not| b$. If x_m is a square of a rational number for some positive integer m, prove that x_1 is also a square of a rational number.

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2 Let ABC be an acute-angled triangle with $AB < AC < BC$ and $c(O, R)$ the circumscribed circle. Let D, E be points in the small arcs AC, AB respectively. Let K be the intersection point of BD, CE and N the second common point of the circumscribed circles of the triangles BKE and CKD. Prove that A, K, N are collinear if and only if K belongs to the symmedian of ABC passing from A.

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3 Let n, m be positive integers such that $n < m$ and $a_1, a_2, ..., a_m$ be different real numbers.
(a) Find all polynomials P with real coefficients and degree at most n such that: $|P(a_i) - P(a_j)| = |a_i - a_j|$ for all $i, j = \{1, 2, ..., m\}$ such that $i < j$.
(b) If $n, m \geq 2$ does there exist a polynomial Q with real coefficients and degree n such that: $|Q(a_i) - Q(a_j)| < |a_i - a_j|$ for all $i, j = \{1, 2, ..., m\}$ such that $i < j$
Edit: See#3

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4 In the plane, there are n points ($n \geq 4$) where no 3 of them are collinear. Let $A(n)$ be the number of parallelograms whose vertices are those points with area 1. Prove the following inequality: $A(n) \leq \frac{n^2-3n}{4}$ for all $n \geq 4$
