III Caucasus Mathematical Olympiad

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by bigant146

– Juniors

– First day

1 Let a, b, c be real numbers, not all of them are equal. Prove that $a+b+c=0$ if and only if $a^2+ab+b^2=b^2+bc+c^2=c^2+ca+a^2$.

2 On a chessboard 8×8, $n>6$ Knights are placed so that for any 6 Knights there are two Knights that attack each other. Find the greatest possible value of n.

3 Suppose that a,b,c are positive integers such that a^b divides b^c, and a^c divides c^b. Prove that a^2 divides bc.

4 By *centroid* of a quadrilateral $PQRS$ we call a common point of two lines through the midpoints of its opposite sides. Suppose that $ABCDEF$ is a hexagon inscribed into the circle Ω centered at O. Let $AB=DE$, and $BC=EF$. Let X,Y, and Z be centroids of $ABDE$, $BCEF$; and $CDFA$, respectively. Prove that O is the orthocenter of triangle XYZ.

– Second day

5 Baron Munhausen discovered the following theorem: "For any positive integers a and b there exists a positive integer n such that an is a perfect square, while bn is a perfect cube". Determine if the statement of Barons theorem is correct.

6 Given a convex quadrilateral $ABCD$ with $\angle BCD=90^\circ$. Let E be the midpoint of AB. Prove that $2EC\leqslant AD+BD$.

7 Given a positive integer $n>1$. In the cells of an $n\times n$ board, marbles are placed one by one. Initially there are no marbles on the board. A marble could be placed in a free cell neighboring (by side) with at least two cells which are still free. Find the greatest possible number of marbles that could be placed on the board according to these rules.

8 Let a,b,c be the lengths of sides of a triangle. Prove the inequality

$$(a+b)\sqrt{ab}+(a+c)\sqrt{ac}+(b+c)\sqrt{bc}\geq (a+b+c)^2/2.$$

– Seniors

– First day

1 A tetrahedron is given. Determine whether it is possible to put some 10 consecutive positive integers at 4 vertices and at 6 midpoints of the edges so that the number at the midpoint of each edge is equal to the arithmetic mean of two numbers at the endpoints of this edge.

2 Let I be the incenter of an acute-angled triangle ABC. Let P, Q, R be points on sides AB, BC, CA respectively, such that $AP = AR$, $BP = BQ$ and $\angle PIQ = \angle BAC$. Prove that $QR \perp AC$.

3 For $2n$ positive integers a matching (i.e. dividing them into n pairs) is called *non-square* if the product of two numbers in each pair is not a perfect square. Prove that if there is a non-square matching, then there are at least $n!$ non-square matchings.
(By $n!$ denote the product $1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.)

4 Morteza places a function $[0, 1] \to [0, 1]$ (that is a function with domain [0,1] and values from [0,1]) in each cell of an $n \times n$ board. Pavel wants to place a function $[0, 1] \to [0, 1]$ to the left of each row and below each column (i.e. to place $2n$ functions in total) so that the following condition holds for any cell in this board:
If h is the function in this cell, f is the function below its column, and g is the function to the left of its row, then $h(x) = f(g(x))$ for all $x \in [0, 1]$.
Prove that Pavel can always fulfil his plan.

– Second day

5 Baron Munhausen discovered the following theorem: "For any positive integers a and b there exists a positive integer n such that an is a perfect cube, while bn is a perfect fifth power". Determine if the statement of Barons theorem is correct.

6 Two graphs G_1 and G_2 of quadratic polynomials intersect at points A and B. Let O be the vertex of G_1. Lines OA and OB intersect G_2 again at points C and D. Prove that CD is parallel to the x-axis.

7 In an acute-angled triangle ABC, the altitudes from A, B, C meet the sides of ABC at A_1, B_1, C_1, and meet the circumcircle of ABC at A_2, B_2, C_2, respectively. Line A_1C_1 intersects the circumcircles of triangles AC_1C_2 and CA_1A_2 at points P and Q ($Q \neq A_1$, $P \neq C_1$). Prove that the circle PQB_1 touches the line AC.

8 In the cells of an 8×8 board, marbles are placed one by one. Initially there are no marbles on the board. A marble could be placed in a free cell neighboring (by side) with at least three cells which are still free. Find the greatest possible number of marbles that could be placed on the board according to these rules.

