Korea National Olympiad 1997

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–	Day 1
1	Let $f(n)$ be the number of ways to express positive integer n as a sum of positive odd integers. Compute $f(n)$. (If the order of odd numbers are different, then it is considered as different expression.)
2	For positive integer n, let $a_n = \sum_{k=0}^{[\frac{n}{2}]} \binom{n-2}{k}(-\frac{1}{4})^k$. Find a_{1997}. (For real x, $[x]$ is defined as largest integer that does not exceeds x.)
3	Let $ABCDEF$ be a convex hexagon such that $AB = BC, CD = DE, EF = FA$. Prove that $\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{3}{2}$ and find when equality holds.
4	For any prime number $p > 2$, and an integer a and b, if $1 + \frac{1}{2^3} + \frac{1}{3^3} + \cdots + \frac{1}{(p-1)^3} = \frac{a}{b}$, prove that a is divisible by p.
–	Day 2
5	Let a, b, c be the side lengths of any triangle $\triangle ABC$ opposite to A, B and C, respectively. Let x, y, z be the length of medians from A, B and C, respectively. If T is the area of $\triangle ABC$, prove that $\frac{a^2}{x} + \frac{b^2}{y} + \frac{c^2}{z} \geq \sqrt{\sqrt{3}T}$
6	Find all polynomial $P(x, y)$ for any reals x, y such that (i) $x^{100} + y^{100} \leq P(x, y) \leq 101(x^{100} + y^{100})$ (ii) $(x - y)P(x, y) = (x - 1)P(x, 1) + (1 - y)P(1, y)$.
7	Let X, Y, Z be the points outside the $\triangle ABC$ such that $\angle BAZ = \angle CAY, \angle CBX = \angle ABZ, \angle ACY = \angle BCX$. Prove that the lines AX, BY, CZ are concurrent.
8	For any positive integers x, y, z and w, prove that x^2, y^2, z^2 and w^2 cannot be four consecutive terms of arithmetic sequence.