Korea National Olympiad 2006

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–	Day 1
1	Given that for reals $a_1, \cdots, a_{2004}$, equation $x^{2006}-2006x^{2005}+a_{2004}x^{2004}+\cdots+a_2x^2+a_1x+1=0$ has 2006 positive real solution, find the maximum possible value of a_1.
2	Alice and Bob are playing "factoring game." On the paper, $270000(=2^4 3^3 5^4)$ is written and each person picks one number from the paper(call it N) and erase N and writes integer X, Y such that $N=XY$ and gcd$(X,Y)\neq 1$. Alice goes first and the person who can no longer make this factoring loses. If two people use optimal strategy, prove that Alice always win.
3	For three positive integers a, b and c, if gcd$(a,b,c)=1$ and $a^2+b^2+c^2=2(ab+bc+ca)$, prove that all of a, b, c is perfect square.
4	On the circle O, six points A, B, C, D, E, F are on the circle counterclockwise. BD is the diameter of the circle and it is perpendicular to CF. Also, lines CF, BE, AD is concurrent. Let M be the foot of altitude from B to AC and let N be the foot of altitude from D to CE. Prove that the area of $\triangle MNC$ is less than half the area of $\Box ACEF$.
–	Day 2
5	Find all positive integers n such that $\phi(n)$ is the fourth power of some prime.
6	Prove that for any positive real numbers x, y and z, $xyz(x+2)(y+2)(z+2)\leq(1+\frac{2(xy+yz+zx)}{3})^3$
7	Points A, B, C, D, E, F is on the circle O. A line ℓ is tangent to O at E is parallel to AC and $DE > EF$. Let P, Q be the intersection of ℓ and BC, CD ,respectively and let R, S be the intersection of ℓ and CF, DF ,respectively. Show that $PQ=RS$ if and only if $QE=ER$.
8	27 students are given a number from 1 to 27. How many ways are there to divide 27 students into 9 groups of 3 with the following condition? (i) The sum of students number in each group is $1 \pmod 3$ (ii) There are no such two students where their numbering differs by 3.