China Girls Math Olympiad 2017

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by CinarArslan, smy2012, Lsway, sqing, Hermitianism, yufanfeng

– Day 1

1 (1) Find all positive integer n such that for any odd integer a, we have $4 \mid a^n - 1$
(2) Find all positive integer n such that for any odd integer a, we have $2^{2017} \mid a^n - 1$

2 Given quadrilateral $ABCD$ such that $\angle BAD + 2\angle BCD = 180^\circ$.
Let E be the intersection of BD and the internal bisector of $\angle BAD$.
The perpendicular bisector of AE intersects CB, CD at X, Y, respectively.
Prove that A, C, X, Y are concyclic.

3 Given $a_i \geq 0, x_i \in \mathbb{R}, (i = 1, 2, \ldots, n)$. Prove that

$$((1 - \sum_{i=1}^{n} a_i \cos x_i)^2 + (1 - \sum_{i=1}^{n} a_i \sin x_i)^2)^2 \geq 4(1 - \sum_{i=1}^{n} a_i)^3$$

4 Partition $\frac{1}{2002}, \frac{1}{2003}, \frac{1}{2004}, \ldots, \frac{1}{2017}$ into two groups. Define A the sum of the numbers in the first group, and B the sum of the numbers in the second group. Find the partition such that $|A - B|$ attains it minimum and explains the reason.

– Day 2

5 Let $0 = x_0 < x_1 < \cdots < x_n = 1$ .Find the largest real number$C$ such that for any positive integer n , we have

$$\sum_{k=1}^{n} x_k^2 (x_k - x_{k-1}) > C$$

6 Given a finite set X, two positive integers n, k, and a map $f : X \to X$. Define $f^{(1)}(x) = f(x), f^{(i+1)}(x) = f^{(i)}(x)$,$i = 1, 2, 3, \ldots$. It is known that for any $x \in X$,$f^{(n)}(x) = x$.
Define m_j the number of $x \in X$ satisfying $f^{(j)}(x) = x$.
Prove that:
(1)$\frac{1}{n} \sum_{j=1}^{n} m_j \sin \frac{2kj\pi}{n} = 0$
(2)$\frac{1}{n} \sum_{j=1}^{n} m_j \cos \frac{2kj\pi}{n}$ is a non-negative integer.

7 This is a very classical problem.
Let the $ABCD$ be a cyclic quadrilateral with circumcircle ω_1.Lines AC and BD intersect at

point E,and lines AD,BC intersect at point F.Circle ω_2 is tangent to segments EB, EC at points M, N respectively,and intersects with circle ω_1 at points Q, R.Lines BC, AD intersect line MN at S, T respectively.Show that Q, R, S, T are concyclic.

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8 Let n be a fixed positive integer. Let

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

be two $n \times n$ tables such that $\{a_{ij}|1 \le i, j \le n\} = \{b_{ij}|1 \le i, j \le n\} = \{k \in N^*|1 \le k \le n^2\}$.
One can perform the following operation on table A: Choose 2 numbers in the same row or in the same column of A, interchange these 2 numbers, and leave the remaining $n^2 - 2$ numbers unchanged. This operation is called a **transposition** of A.
Find, with proof, the smallest positive integer m such that for any tables A and B, one can perform at most m transpositions such that the resulting table of A is B.
