Final Round - Korea 2018

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Day 1 March 24th

1 Find all integers of the form $\frac{m-6n}{m+2n}$ where m, n are nonzero rational numbers satisfying $m^3 = (27n^2+1)(m+2n)$.

2 Triangle ABC satisfies $\angle ABC < \angle BCA < \angle CAB < 90^\circ$. O is the circumcenter of triangle ABC, and K is the reflection of O in BC. D, E is the foot of perpendicular line from K to line AB, AC, respectively. Line DE meets BC at P, and a circle with diameter AK meets the circumcircle of triangle ABC at $Q(\neq A)$. If PQ cuts the perpendicular bisector of BC at S, then prove that S lies on the circle with diameter AK.

3 For 31 years, n (¿6) tennis players have records of wins. It turns out that for every two players, there is a third player who has won over them before. Prove that for every integer k, l such that $2^{2^k+1} - 1 > n, 1 < l < 2k+1$, there exist l players ($A_1, A_2, ..., A_l$) such that every player A_{i+1} won over A_i. (A_{l+1} is same as A_1)

Day 2 March 25th

4 Triangle ABC satisfies $\angle C = 90^\circ$. A circle passing A, B meets segment AC at $G(\neq A, C)$ and it meets segment BC at point $D(\neq B)$. Segment AD cuts segment BG at H, and let l, the perpendicular bisector of segment AD, cuts the perpendicular bisector of segment AB at point E. A line passing D is perpendicular to DE and cuts l at point F. If the circumcircle of triangle CFH cuts AC, BC at $P(\neq C), Q(\neq C)$ respectively, then prove that PQ is perpendicular to FH.

5 Determine whether or not two polynomials P, Q with degree no less than 2018 and with integer coefficients exist such that

$$P(Q(x)) = 3Q(P(x)) + 1$$

for all real numbers x.

6 Twenty ants live on the faces of an icosahedron, one ant on each side, where the icosahedron have each side with length 1. Each ant moves in a counterclockwise direction on each face, along the side/edges. The speed of each ant must be no less than 1 always. Also, if two ants meet, they should meet at the vertex of the icosahedron. If five ants meet at the same time at a vertex, we call that a *collision*. Can the ants move forever, in a way that no *collision* occurs?