Turkey Team Selection Test 2018

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Day 1 24 March 2018

1 Prove that, for all integers a, b, there exists a positive integer n, such that the number n^2+an+b has at least 2018 different prime divisors.

2 Find all $f : \mathbb{R} \to \mathbb{R}$ surjective functions such that

$$f(xf(y)+y^2) = f((x+y)^2) - xf(x)$$

for all real numbers x, y.

3 A Retired Linguist (R.L.) writes in the first move a word consisting of n letters, which are all different. In each move, he determines the maximum i, such that the word obtained by reversing the first i letters of the last word hasn't been written before, and writes this new word. Prove that R.L. can make $n!$ moves.

Day 2 25 March 2018

4 In a non-isosceles acute triangle ABC, D is the midpoint of the edge $[BC]$. The points E and F lie on $[AC]$ and $[AB]$, respectively, and the circumcircles of CDE and AEF intersect in P on $[AD]$. The angle bisector from P in triangle EFP intersects EF in Q. Prove that the tangent line to the circumcirle of AQP at A is perpendicular to BC.

5 We say that a group of 25 students is a *team* if any two students in this group are friends. It is known that in the school any student belongs to at least one team but if any two students end their friendships at least one student does not belong to any team. We say that a team is *special* if at least one student of the team has no friend outside of this team. Show that any two friends belong to some special team.

6 $a_0, a_1, \ldots, a_{100}$ and $b_1, b_2, \ldots, b_{100}$ are sequences of real numbers, for which the property holds: for all $n = 0, 1, \ldots, 99$, either

$$a_{n+1} = \frac{a_n}{2} \quad \text{and} \quad b_{n+1} = \frac{1}{2} - a_n,$$

or

$$a_{n+1} = 2a_n^2 \quad \text{and} \quad b_{n+1} = a_n.$$

Given $a_{100} \leq a_0$, what is the maximal value of $b_1 + b_2 + \cdots + b_{100}$?

Day 3 26 March 2018

7 For integers a, b, call the lattice point with coordinates (a, b) **basic** if $gcd(a, b) = 1$. A graph takes the basic points as vertices and the edges are drawn in such way: There is an edge between (a_1, b_1) and (a_2, b_2) if and only if $2a_1 = 2a_2 \in \{b_1 - b_2, b_2 - b_1\}$ or $2b_1 = 2b_2 \in \{a_1 - a_2, a_2 - a_1\}$. Some of the edges will be erased, such that the remaining graph is a forest. At least how many edges must be erased to obtain this forest? At least how many trees exist in such a forest?

8 For integers $m \geq 3$, n and $x_1, x_2, \dots, x_m$ if $x_{i+1} - x_i \equiv x_i - x_{i-1} (mod n)$ for every $2 \leq i \leq m-1$, we say that the m-tuple $(x_1, x_2, \dots, x_m)$ is an arithmetic sequence in $(mod n)$. Let $p \geq 5$ be a prime number and $1 < a < p-1$ be an integer. Let $a_1, a_2, \dots, a_k$ be the set of all possible remainders when positive powers of a are divided by p. Show that if a permutation of $a_1, a_2, \dots, a_k$ is an arithmetic sequence in $(mod p)$, then $k = p-1$.

9 For a triangle T and a line d, if the feet of perpendicular lines from a point in the plane to the edges of T all lie on d, say d focuses T. If the set of lines focusing T_1 and the set of lines focusing T_2 are the same, say T_1 and T_2 are equivalent. Prove that, for any triangle in the plane, there exists exactly one equilateral triangle which is equivalent to it.