Canada National Olympiad 2018

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1 Consider an arrangement of tokens in the plane, not necessarily at distinct points. We are allowed to apply a sequence of moves of the following kind: select a pair of tokens at points A and B and move both of them to the midpoint of A and B.

We say that an arrangement of n tokens is *collapsible* if it is possible to end up with all n tokens at the same point after a finite number of moves. Prove that every arrangement of n tokens is collapsible if and only if n is a power of 2.

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2 Let five points on a circle be labelled A, B, C, D, and E in clockwise order. Assume $AE = DE$ and let P be the intersection of AC and BD. Let Q be the point on the line through A and B such that A is between B and Q and $AQ = DP$ Similarly, let R be the point on the line through C and D such that D is between C and R and $DR = AP$. Prove that PE is perpendicular to QR.

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3 Two positive integers a and b are prime-related if $a = pb$ or $b = pa$ for some prime p. Find all positive integers n, such that n has at least three divisors, and all the divisors can be arranged without repetition in a circle so that any two adjacent divisors are prime-related.

Note that 1 and n are included as divisors.

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4 Find all polynomials $p(x)$ with real coefficients that have the following property: there exists a polynomial $q(x)$ with real coefficients such that

$$p(1) + p(2) + p(3) + \cdots + p(n) = p(n)q(n)$$

for all positive integers n.

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5 Let k be a given even positive integer. Sarah first picks a positive integer N greater than 1 and proceeds to alter it as follows: every minute, she chooses a prime divisor p of the current value of N, and multiplies the current N by $p^k - p^{-1}$ to produce the next value of N. Prove that there are infinitely many even positive integers k such that, no matter what choices Sarah makes, her number N will at some point be divisible by 2018.
