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Serbia National Math Olympiad 2018

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– Day 1

1 Let $\triangle ABC$ be a triangle with incenter I. Points P and Q are chosen on segmets BI and CI such that $2\angle PAQ = \angle BAC$. If D is the touch point of incircle and side BC prove that $\angle PDQ = 90$.

2 Let $n > 1$ be an integer. Call a number beautiful if its square leaves an odd remainder upon divison by n. Prove that the number of consecutive beautiful numbers is less or equal to $1 + \lfloor \sqrt{3n} \rfloor$.

3 Let n be a positive integer. There are given n lines such that no two are parallel and no three meet at a single point.
a) Prove that there exists a line such that the number of intersection points of these n lines on both of its sides is at least

$$\left\lfloor \frac{(n-1)(n-2)}{10} \right\rfloor.$$

Notice that the points on the line are not counted.
b) Find all n for which there exists a configurations where the equality is achieved.

– Day 2

4 Prove that there exists a uniqe $P(x)$ polynomial with real coefficients such that
$xy - x - y | (x+y)^{1000} - P(x) - P(y)$ for all real x, y.

5 Let $a, b > 1$ be odd positive integers. A board with a rows and b columns without fields $(2,1)$, $(a-2, b)$ and (a, b) is tiled with 2×2 squares and 2×1 dominoes (that can be rotated). Prove that the number of dominoes is at least

$$\frac{3}{2}(a+b) - 6.$$

6 For each positive integer k, let n_k be the smallest positive integer such that there exists a finite set A of integers satisfy the following properties:

-For every $a \in A$, there exists $x, y \in A$ (not necessary distinct) that

$$n_k \mid a - x - y$$

-There's no subset B of A that $|B| \leq k$ and

$$n_k \mid \sum_{b \in B} b.$$

Show that for all positive integers $k \geq 3$, we've

$$n_k < \left(\frac{13}{8}\right)^{k+2}.$$
