Sharygin Geometry Olympiad 2018

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– **First (Correspondence) Round**

– Grade 8

1 Three circles lie inside a square. Each of them touches externally two remaining circles. Also each circle touches two sides of the square. Prove that two of these circles are congruent.

2 A cyclic quadrilateral $ABCD$ is given. The lines AB and DC meet at point E, and the lines BC and AD meet at point F. Let I be the incenter of triangle AED, and a ray with origin F be perpendicular to the bisector of angle AID. In which ratio this ray dissects the angle AFB?

3 Let AL be the bisector of triangle ABC, D be its midpoint, and E be the projection of D to AB. It is known that $AC = 3AE$. Prove that CEL is an isosceles triangle.

4 Let $ABCD$ be a cyclic quadrilateral. A point P moves along the arc AD which does not contain B and C. A fixed line l, perpendicular to BC, meets the rays BP, CP at points B_0, C_0 respectively. Prove that the tangent at P to the circumcircle of triangle PB_0C_0 passes through some fixed point.

– Grades 8-9

5 The vertex C of equilateral triangles ABC and CDE lies on the segment AE, and the vertices B and D lie on the same side with respect to this segment. The circumcircles of these triangles centered at O_1 and O_2 meet for the second time at point F. The lines O_1O_2 and AD meet at point K. Prove that $AK = BF$.

6 Let CH be the altitude of a right-angled triangle ABC ($\angle C = 90^\circ$) with $BC = 2AC$. Let O_1, O_2 and O be the incenters of triangles ACH, BCH and ABC respectively, and H_1, H_2, H_0 be the projections of O_1, O_2, O respectively to AB. Prove that $H_1H = HH_0 = H_0H_2$.

7 Let E be a common point of circles ω_1 and ω_2. Let AB be a common tangent to these circles, and CD be a line parallel to AB, such that A and C lie on ω_1, B and D lie on ω_2. The circles ABE and CDE meet for the second time at point F. Prove that F bisects one of arcs CD of circle CDE.

8 Restore a triangle ABC by the Nagel point, the vertex B and the foot of the altitude from this vertex.

9 A square is inscribed into an acute-angled triangle: two vertices of this square lie on the same side of the triangle and two remaining vertices lies on two remaining sides. Two similar squares are constructed for the remaining sides. Prove that three segments congruent to the sides of these squares can be the sides of an acute-angled triangle.

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10 In the plane, 2018 points are given such that all distances between them are different. For each point, mark the closest one of the remaining points. What is the minimal number of marked points?

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11 Let I be the incenter of a nonisosceles triangle ABC. Prove that there exists a unique pair of points M, N lying on the sides AC, BC respectively, such that $\angle AIM = \angle BIN$ and $MN||AB$.

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12 Let BD be the external bisector of a triangle ABC with $AB > BC$; K and K_1 be the touching points of side AC with the incircle and the excircle centered at I and I_1 respectively. The lines BK and DI_1 meet at point X, and the lines BK_1 and DI meet at point Y. Prove that $XY \perp AC$.

– Grades 9-11

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13 Let $ABCD$ be a cyclic quadrilateral, and M, N be the midpoints of arcs AB and CD respectively. Prove that MN bisects the segment between the incenters of triangles ABC and ADC.

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14 Let ABC be a right-angled triangle with $\angle C = 90°$, K, L, M be the midpoints of sides AB, BC, CA respectively, and N be a point of side AB. The line CN meets KM and KL at points P and Q respectively. Points S, T lying on AC and BC respectively are such that $APQS$ and $BPQT$ are cyclic quadrilaterals. Prove that

a) if CN is a bisector, then CN, ML and ST concur;

b) if CN is an altitude, then ST bisects ML.

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15 The altitudes AH_1, BH_2, CH_3 of an acute-angled triangle ABC meet at point H. Points P and Q are the reflections of H_2 and H_3 with respect to H. The circumcircle of triangle PH_1Q meets for the second time BH_2 and CH_3 at points R and S. Prove that RS is a medial line of triangle ABC.

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16 Let ABC be a triangle with $AB < BC$. The bisector of angle C meets the line parallel to AC and passing through B, at point P. The tangent at B to the circumcircle of ABC meets this bisector at point R. Let R' be the reflection of R with respect to AB. Prove that $\angle R'PB = \angle RPA$.

– Grade 10-11

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17 Let each of circles α, β, γ touches two remaining circles externally, and all of them touche a circle Ω internally at points A_1, B_1, C_1 respectively. The common internal tangent to α and β meets the arc A_1B_1 not containing C_1 at point C_2. Points A_2, B_2 are defined similarly. Prove that the lines A_1A_2, B_1B_2, C_1C_2 concur.

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18 Let C_1, A_1, B_1 be points on sides AB, BC, CA of triangle ABC, such that AA_1, BB_1, CC_1 concur. The rays B_1A_1 and B_1C_1 meet the circumcircle of the triangle at points A_2 and C_2 respectively. Prove that A, C, the common point of A_2C_2 and BB_1 and the midpoint of A_2C_2 are concyclic.

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19 Let a triangle ABC be given. On a ruler three segment congruent to the sides of this triangle are marked. Using this ruler construct the orthocenter of the triangle formed by the tangency points of the sides of ABC with its incircle.

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20 Let the incircle of a nonisosceles triangle ABC touch AB, AC and BC at points D, E and F respectively. The corresponding excircle touches the side BC at point N. Let T be the common point of AN and the incircle, closest to N, and K be the common point of DE and FT. Prove that $AK||BC$.

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21 In the plane a line l and a point A outside it are given. Find the locus of the incenters of acute-angled triangles having a vertex A and an opposite side lying on l.

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22 Six circles of unit radius lie in the plane so that the distance between the centers of any two of them is greater than d. What is the least value of d such that there always exists a straight line which does not intersect any of the circles and separates the circles into two groups of three?

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23 The plane is divided into convex heptagons with diameters less than 1. Prove that an arbitrary disc with radius 200 intersects most than a billion of them.

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24 A crystal of pyrite is a parallelepiped with dashed faces. The dashes on any two adjacent faces are perpendicular. Does there exist a convex polytope with the number of faces not equal to 6, such that its faces can be dashed in such a manner?

– **Final Round**

– Grade 8

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1 The incircle of a right-angled triangle ABC ($\angle C = 90^\circ$) touches BC at point K. Prove that the chord of the incircle cut by line AK is twice as large as the distance from C to that line.

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2 A rectangle $ABCD$ and its circumcircle are given. Let E be an arbitrary point on the minor arc BC. The tangent to the circle at B meets CE at point G. The segments AE and BD meet at point K. Prove that GK and AD are perpendicular.

3 Let ABC be a triangle with $\angle A = 60^\circ$, and AA', BB', CC' be its internal angle bisectors. Prove that $\angle B'A'C' \leq 60^\circ$.

4 Find all sets of six points in the plane, no three collinear, such that if we partition the set into two sets, then the obtained triangles are congruent.

5 The side AB of a square $ABCD$ is the base of an isosceles triangle ABE such that $AE = BE$ lying outside the square. Let M be the midpoint of AE, O be the intersection of AC and BD. K is the intersection of OM and ED. Prove that $EK = KO$.

6 Suppose $ABCD$ and $A_1B_1C_1D_1$ be quadrilaterals with corresponding angles equal. Also $AB = A_1B_1$, $AC = A_1C_1$, $BD = B_1D_1$. Are the quadrilaterals necessarily congruent?

7 Let ω_1, ω_2 be two circles centered at O_1 and O_2 and lying outside each other. Points C_1 and C_2 lie on these circles in the same semi plane with respect to O_1O_2. The ray O_1C_1 meets ω_2 at A_2, B_2 and O_2C_2 meets ω_1 at A_1, B_1. Prove that $\angle A_1O_1B_1 = \angle A_2O_2B_2$ if and only if $C_1C_2 || O_1O_2$.

8 Let I be the incenter of fixed triangle ABC, and D be an arbitrary point on BC. The perpendicular bisector of AD meets BI, CI at F and E respectively. Find the locus of orthocenters of $\triangle IEF$ as D varies.

– Grade 9

1 Let M be the midpoint of AB in a right angled triangle ABC with $\angle C = 90^\circ$. A circle passing through C and M meets segments BC, AC at P, Q respectively. Let c_1, c_2 be the circles with centers P, Q and radii BP, AQ respectively. Prove that c_1, c_2 and the circumcircle of ABC are concurrent.

2 A triangle ABC is given. A circle γ centered at A meets segments AB and AC. The common chord of γ and the circumcircle of ABC meets AB and AC at X and Y, respectively. The segments CX and BY meet γ at point S and T, respectively. The circumcircles of triangles ACT and BAS meet at points A and P. Prove that CX, BY and AP concur.

3 The vertices of a triangle DEF lie on different sides of a triangle ABC. The lengths of the tangents from the incenter of DEF to the excircles of ABC are equal. Prove that $4S_{DEF} \geq S_{ABC}$.

[i]Note: By S_{XYZ} we denote the area of triangle XYZ.[/i]

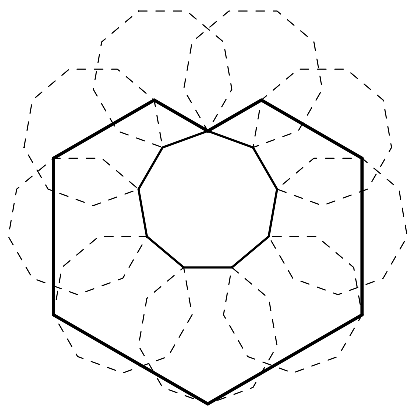
4 Let BC be a fixed chord of a circle ω. Let A be a variable point on the major arc BC of ω. Let H be the orthocenter of ABC. The points D, E lie on AB, AC such that H is the midpoint of DE. O_A is the circumcenter of ADE. Prove that as A varies, O_A lies on a fixed circle.

5 Let $ABCD$ be a cyclic quadrilateral, BL and CN be the internal angle bisectors in triangles ABD and ACD respectively. The circumcircles of triangles ABL and CDN meet at points P and Q. Prove that the line PQ passes through the midpoint of the arc AD not containing B.

6 Let $ABCD$ be a circumscribed quadrilateral. Prove that the common point of the diagonals, the incenter of triangle ABC and the centre of excircle of triangle CDA touching the side AC are collinear.

7 Let B_1, C_1 be the midpoints of sides AC, AB of a triangle ABC respectively. The tangents to the circumcircle at B and C meet the rays CC_1, BB_1 at points K and L respectively. Prove that $\angle BAK = \angle CAL$.

8 Consider a fixed regular n-gon of unit side. When a second regular n-gon of unit size rolls around the first one, one of its vertices successively pinpoints the vertices of a closed broken line κ as in the figure.



Let A be the area of a regular n-gon of unit side, and let B be the area of a regular n-gon of unit circumradius. Prove that the area enclosed by κ equals $6A - 2B$.

– Grade 10

1 The altitudes AH, CH of an acute-angled triangle ABC meet the internal bisector of angle B at points L_1, P_1, and the external bisector of this angle at points L_2, P_2. Prove that the orthocenters of triangles HL_1P_1, HL_2P_2 and the vertex B are collinear.

2 A fixed circle ω is inscribed into an angle with vertex C. An arbitrary circle passing through C, touches ω externally and meets the sides of the angle at points A and B. Prove that the perimeters of all triangles ABC are equal.

3 A cyclic n-gon is given. The midpoints of all its sides are concyclic. The sides of the n-gon cut n arcs of this circle lying outside the n-gon. Prove that these arcs can be coloured red and blue in such a way that the sum of the lengths of the red arcs is equal to the sum of the lengths of the blue arcs.

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4 We say that a finite set S of red and green points in the plane is *separable* if there exists a triangle δ such that all points of one colour lie strictly inside δ and all points of the other colour lie strictly outside of δ. Let A be a finite set of red and green points in the plane, in general position. Is it always true that if every 1000 points in A form a separable set then A is also separable?

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5 Let ω be the incircle of a triangle ABC. The line passing though the incenter I and parallel to BC meets ω at A_b and A_c (A_b lies in the same semi plane with respect to AI as B). The lines BA_b and CA_c meet at A_1. The points B_1 and C_1 are defined similarly. prove that AA_1, BB_1, CC_1 concur.

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6 Let ω be the circumcircle of ABC, and KL be the diameter of ω passing through M midpoint of AB (K, C lies on different sides of AB). A circle passing through L and M meets CK at points P and Q (Q lies on KP). Let LQ meet the circumcircle of KMQ again at R. Prove that $APBR$ is cyclic.

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7 A convex quadrilateral $ABCD$ is circumscribed about a circle of radius r. What is the maximum value of $\frac{1}{AC^2} + \frac{1}{BD^2}$?

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8 Two triangles ABC and $A'B'C'$ are given. The lines AB and $A'B'$ meet at C_1 and the lines parallel to them and passing through C and C' meet at C_2. The points A_1, A_2, B_1, B_2 are defined similarly. Prove that A_1A_2, B_1B_2, C_1C_1 are either parallel or concurrent.
