Moldova Team Selection Test 2018

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– Day 1

1 Let $x,y,z \in \mathbb{Q}$,such that $(x+y+z)^3=9(x^2y+y^2z+z^2x)$. Prove that $x=y=z$

2 The sequence $(a_n)_{n\in \mathbb{N}}$ is defined recursively as $a_0=a_1=1$, $a_{n+2}=5a_{n+1}-a_n-1$, $\forall n\in \mathbb{N}$
Prove that

$$a_n \mid a_{n+1}^2+a_{n+1}+1$$

for any $n\in \mathbb{N}$

3 Let O be the circumcenter of an acute triangle ABC. Line OA intersects the altitudes of ABC through B and C at P and Q, respectively. The altitudes meet at H. Prove that the circumcenter of triangle PQH lies on a median of triangle ABC.

4 A pupil is writing on a board positive integers $x_0,x_1,x_2,x_3...$ after the following algorithm which implies arithmetic progression $3,5,7,9...$.Each term of rank $k\geq 2$ is a difference between the product of the last number on the board and the term of arithmetic progression of rank k and the last but one term on the bord with the sum of the terms of the arithemtic progression with ranks less than k.If $x_0=0$ and $x_1=1$ find x_n according to n.

– Day 2

5 Let $n,\in \mathbb{N}^*,n\geq 3$
a) Prove that the polynomial $f(x)=\frac{X^{2^n-1}-1}{X-1}-X^n$ has a divisor of form X^p+1 where $p\in \mathbb{N}^*$
b) Show that for $n=7$ the polynomial $f(X)$ has three divisors with integer coefficients .

6 Let a,b,c be positive real numbers such that $a+b+c=3$. Show that

$$\frac{a}{1+b^2}+\frac{b}{1+c^2}+\frac{c}{1+a^2}\geq \frac{3}{2}.$$

7 Let the triangle ABC with $m(\angle ABC)=60^\circ$ and $m(\angle BAC)=40^\circ$. Points D and E are on the sides (AB) and (AC) such that $m(\angle DCB)=70^\circ$ and $m(\angle EBC)=40^\circ$. BE and CD intersect in F . Prove that BC and AF are perpendicular.

8 Let the set $A = 1, 2, 3, \ldots, 48n + 24$, where $n \in \mathbb{N}^*$. Prove that there exist a subset B of A with $24n + 12$ elements with the property : the sum of the squares of the elements of the set B is equal to the sum of the squares of the elements of the set $A \quad B$.

– Day 3

9 The positive integers a and b satisfy the sistem $\begin{cases} a_{10} + b_{10} = a \\ a_{11} + b_{11} = b \end{cases}$ where $a_1 < a_2 < \ldots$ and $b_1 < b_2 < \ldots$ are the positive divisors of a and b .
Find a and b .

10 The positive real numbers a, b, c, d satisfy the equality $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} + \frac{1}{d+1} = 3$. Prove the inequality $\sqrt[3]{abc} + \sqrt[3]{bcd} + \sqrt[3]{cda} + \sqrt[3]{dab} \leq \frac{4}{3}$.

11 Let Ω be the circumcincle of the quadrilateral $ABCD$, and E the intersection point of the diagonals AC and BD . A line passing through E intersects AB and BC in points P and Q . A circle ,that is passing through point D , is tangent to the line PQ in point E and intersects Ω in point R , different from D . Prove that the points $B, P, Q,$ and R are concyclic .

12 Let $p > 3$ is a prime number and $k = \lfloor \frac{2p}{3} \rfloor$. Prove that

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

is divisible by p^2.