Romania National Olympiad 2018

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–	Grade 9
1	Prove that if in a triangle the orthocenter, the centroid and the incenter are collinear, then the triangle is isosceles.
2	Let $a, b, c \geq 0$ and $a+b+c=3$. Prove that $$\frac{a}{1+b}+\frac{b}{1+c}+\frac{c}{1+a} \geq \frac{1}{1+b}+\frac{1}{1+c}+\frac{1}{1+a}$$
3	Let $f, g : \mathbb{R} \to \mathbb{R}$ be two quadratics such that, for any real number r, if $f(r)$ is an integer, then $g(r)$ is also an integer. Prove that there are two integers m and n such that $$g(x)=mf(x)+n,\ \forall x \in \mathbb{R}$$
4	Let $n \in \mathbb{N}^*$ and consider a circle of length $6n$ along with $3n$ points on the circle which divide it into $3n$ arcs: n of them have length 1, some other n have length 2 and the remaining n have length 3. Prove that among these points there must be two such that the line that connects them passes through the center of the circle.
–	Grade 10
1	Let $n \in \mathbb{N}_{\geq 2}$ and $a_1, a_2, \ldots, a_n \in (1, \infty)$. Prove that $f : [0, \infty) \to \mathbb{R}$ with $$f(x)=(a_1a_2...a_n)^x-a_1^x-a_2^x-...-a_n^x$$ is a strictly increasing function.
2	Let ABC be a triangle, O its circumcenter and $R=1$ its circumradius. Let G_1, G_2, G_3 be the centroids of the triangles OBC, OAC and OAB. Prove that the triangle ABC is equilateral if and only if $$AG_1+BG_2+CG_3=4$$

3 Let $n \in \mathbb{N}_{\geq 2}$. Prove that for any complex numbers $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, the following statements are equivalent:
a) $\sum_{k=1}^{n} |z-a_k|^2 \leq \sum_{k=1}^{n} |z-b_k|^2$, $\forall z \in \mathbb{C}$.
b) $\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k$ and $\sum_{k=1}^{n} |a_k|^2 \leq \sum_{k=1}^{n} |b_k|^2$.

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4 Let $n \in \mathbb{N}_{\geq 2}$. For any real numbers $a_1, a_2, ..., a_n$ denote $S_0 = 1$ and for $1 \leq k \leq n$ denote

$$S_k = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} a_{i_1} a_{i_2} \ldots a_{i_k}$$

Find the number of $n-$tuples $(a_1, a_2, \ldots a_n)$ such that

$$(S_n - S_{n-2} + S_{n-4} - \ldots)^2 + (S_{n-1} - S_{n-3} + S_{n-5} - \ldots)^2 = 2^n S_n.$$

– Grade 11

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1 Let $n \geq 2$ be a positive integer and, for all vectors with integer entries

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

let $\delta(X) \geq 0$ be the greatest common divisor of $x_1, x_2, \ldots, x_n$. Also, consider $A \in \mathcal{M}_n(\mathbb{Z})$. Prove that the following statements are equivalent: **i)** $|\det A| = 1$ **ii)** $\delta(AX) = \delta(X)$, for all vectors $X \in \mathcal{M}_{n,1}(\mathbb{Z})$.

Romeo Raicu

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2 Let $x > 0$. Prove that

$$2^{-x} + 2^{-1/x} \leq 1.$$

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3 Let $f : \mathbb{R} \to \mathbb{R}$ be a function with the intermediate value property. If f is injective on $\mathbb{R} \setminus \mathbb{Q}$, prove that f is continuous on $\mathbb{R}$.

Julieta R. Vergulescu

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4 Let n be an integer with $n \geq 2$ and let $A \in \mathcal{M}_n(\mathbb{C})$ such that $\operatorname{rank} A \neq \operatorname{rank} A^2$. Prove that there exists a nonzero matrix $B \in \mathcal{M}_n(\mathbb{C})$ such that

$$AB = BA = B^2 = 0$$

Cornel Delasava

– Grade 12

1 Let A be a finite ring and $a, b \in A$, such that $(ab-1)b = 0$. Prove that $b(ab-1) = 0$.

2 Let $\mathcal{F}$ be the set of continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$e^{f(x)} + f(x) \geq x + 1,\ \forall x \in \mathbb{R}$$

For $f \in \mathcal{F}$, let

$$I(f) = \int_0^e f(x)dx$$

Determine $\min_{f \in \mathcal{F}} I(f)$.

Liviu Vlaicu

3 Let $f : [a, b] \to \mathbb{R}$ be an integrable function and $(a_n) \subset \mathbb{R}$ such that $a_n \to 0$. **a)** If $A = \{m \cdot a_n \mid m, n \in \mathbb{N}^*\}$, prove that every open interval of strictly positive real numbers contains elements from A. **b)** If, for any $n \in \mathbb{N}^*$ and for any $x, y \in [a, b]$ with $|x-y| = a_n$, the inequality $\left|\int_x^y f(t)dt\right| \leq |x-y|$ is true, prove that

$$\left|\int_x^y f(t)dt\right| \leq |x-y|,\ \forall x, y \in [a, b]$$

Nicolae Bourbacut

4 For any $k \in \mathbb{Z}$, define

$$F_k = X^4 + 2(1-k)X^2 + (1+k)^2.$$

Find all values $k \in \mathbb{Z}$ such that F_k is irreducible over $\mathbb{Z}$ and reducible over $\mathbb{Z}_p$, for any prime p.

Marius Vladoiu