Iran Team Selection Test 2018

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Test 1 Day 1

1 Let $A_1,A_2,...,A_k$ be the subsets of $\{1,2,3,...,n\}$ such that for all $1\leq i,j\leq k$:$A_i\cap A_j\neq \varnothing$. Prove that there are n distinct positive integers $x_1,x_2,...,x_n$ such that for each $1\leq j\leq k$:

$$lcm_{i\in A_j}\{x_i\}>lcm_{i\notin A_j}\{x_i\}$$

Proposed by Morteza Saghafian, Mahyar Sefidgaran

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2 Determine the least real number k such that the inequality

$$\left(\frac{2a}{a-b}\right)^2+\left(\frac{2b}{b-c}\right)^2+\left(\frac{2c}{c-a}\right)^2+k \geq 4\left(\frac{2a}{a-b}+\frac{2b}{b-c}+\frac{2c}{c-a}\right)$$

holds for all real numbers a,b,c.

Proposed by Mohammad Jafari

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3 In triangle ABC let M be the midpoint of BC. Let ω be a circle inside of ABC and is tangent to AB,AC at E,F, respectively. The tangents from M to ω meet ω at P,Q such that P and B lie on the same side of AM. Let $X \equiv PM \cap BF$ and $Y \equiv QM \cap CE$. If $2PM=BC$ prove that XY is tangent to ω.

Proposed by Iman Maghsoudi

Test 1 Day 2

4 Let ABC be a triangle ($\angle A\neq 90^\circ$). BE,CF are the altitudes of the triangle. The bisector of $\angle A$ intersects EF,BC at M,N. Let P be a point such that $MP\perp EF$ and $NP\perp BC$. Prove that AP passes through the midpoint of BC.

Proposed by Iman Maghsoudi, Hooman Fattahi

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5 Prove that for each positive integer m, one can find m consecutive positive integers like n such that the following phrase doesn't be a perfect power:

$$\left(1^3+2018^3\right)\left(2^3+2018^3\right)\cdots\left(n^3+2018^3\right)$$

Proposed by Navid Safaei

6 A simple graph is called "divisibility", if it's possible to put distinct numbers on its vertices such that there is an edge between two vertices if and only if number of one of its vertices is divisible by another one.

A simple graph is called "permutationary", if it's possible to put numbers $1, 2, ..., n$ on its vertices and there is a permutation π such that there is an edge between vertices i, j if and only if $i > j$ and $\pi(i) < \pi(j)$ (it's not directed!)

Prove that a simple graph is permutationary if and only if its complement and itself are divisibility.

Proposed by Morteza Saghafian

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Test 2 Day 1

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1 Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the following conditions:
a. $x + f(y + f(x)) = y + f(x + f(y)) \quad \forall x, y \in \mathbb{R}$
b. The set $I = \left\{ \frac{f(x)-f(y)}{x-y} \mid x, y \in \mathbb{R}, x \neq y \right\}$ is an interval.

Proposed by Navid Safaei

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2 Mojtaba and Hooman are playing a game. Initially Mojtaba draws 2018 vectors with zero sum. Then in each turn, starting with Mojtaba, the player takes a vector and puts it on the plane. After the first move, the players must put their vector next to the previous vector (the beginning of the vector must lie on the end of the previous vector).
At last, there will be a closed polygon. If this polygon is not self-intersecting, Mojtaba wins. Otherwise Hooman. Who has the winning strategy?

Proposed by Mahyar Sefidgaran, Jafar Namdar

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3 Let $a_1, a_2, a_3, \cdots$ be an infinite sequence of distinct integers. Prove that there are infinitely many primes p that distinct positive integers i, j, k can be found such that $p \mid a_i a_j a_k - 1$.

Proposed by Mohsen Jamali

Test 2 Day 2

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4 Call a positive integer "useful but not optimized " (!), if it can be written as a sum of distinct powers of 3 and powers of 5.
Prove that there exist infinitely many positive integers which they are not "useful but not optimized".
(e.g. $37 = (3^0 + 3^1 + 3^3) + (5^0 + 5^1)$ is a " useful but not optimized" number)

Proposed by Mohsen Jamali

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5 Let ω be the circumcircle of isosceles triangle ABC ($AB = AC$). Points P and Q lie on ω and BC respectively such that $AP = AQ$ .AP and BC intersect at R. Prove that the tangents from B and C to the incircle of $\triangle AQR$ (different fromBC) are concurrent on ω.

Proposed by Ali Zamani, Hooman Fattahi

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6 $a_1, a_2, \dots, a_n$ is a sequence of positive integers that has at least $\frac{2n}{3} + 1$ distinct numbers and each positive integer has occurred at most three times in it. Prove that there exists a permutation $b_1, b_2, \dots, b_n$ of a_i's such that all the n sums $b_i + b_{i+1}$ are distinct ($1 \le i \le n$, $b_{n+1} \equiv b_1$)

Proposed by Mohsen Jamali

Test 3 Day 1

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1 Two circles $\omega_1(O)$ and ω_2 intersect each other at A, B ,and O lies on ω_2. Let S be a point on AB such that $OS \perp AB$. Line OS intersects ω_2 at P (other than O). The bisector of $A\hat{S}P$ intersects ω_1 at L (A and L are on the same side of the line OP). Let$K$ be a point on ω_2 such that $PS = PK$ (A and K are on the same side of the line OP). Prove that $SL = KL$.

Proposed by Ali Zamani

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2 Find the maximum possible value of k for which there exist distinct reals $x_1, x_2, \dots, x_k$ greater than 1 such that for all $1 \le i, j \le k$,

$$x_i^{\lfloor x_j \rfloor} = x_j^{\lfloor x_i \rfloor}.$$

Proposed by Morteza Saghafian

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3 $n > 1$ and distinct positive integers $a_1, a_2, \dots, a_{n+1}$ are given. Does there exist a polynomial $p(x) \in \mathbb{Z}[x]$ of degree $\le n$ that satisfies the following conditions?
a. $\forall_{1 \le i < j \le n+1} : \gcd(p(a_i), p(a_j)) > 1$
b. $\forall_{1 \le i < j < k \le n+1} : \gcd(p(a_i), p(a_j), p(a_k)) = 1$

Proposed by Mojtaba Zare

Test 3 Day 2

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4 We say distinct positive integers $a_1, a_2, \dots, a_n$ are "good" if their sum is equal to the sum of all pairwise $\gcd$'s among them. Prove that there are infinitely many n s such that n good numbers exist.

Proposed by Morteza Saghafian

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5 $2n - 1$ distinct positive real numbers with sum S are given. Prove that there are at least $\binom{2n-2}{n-1}$ different ways to choose n numbers among them such that their sum is at least $\frac{S}{2}$.

Proposed by Amirhossein Gorzi

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6 Consider quadrilateral $ABCD$ inscribed in circle ω. $P \equiv AC \cap BD$. E, F lie on sides AB, CD respectively such that $A\hat{P}E = D\hat{P}F$. Circles ω_1, ω_2 are tangent to ω at X, Y respectively and also both tangent to the circumcircle of $\triangle PEF$ at P. Prove that:

$$\frac{EX}{EY} = \frac{FX}{FY}$$

Proposed by Ali Zamani
