Czech And Slovak Mathematical Olympiad, Round III, Category A 2018

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by byk7

1 In a group of people, there are some mutually friendly pairs. For positive integer $k \geq 3$ we say the group is k-great, if every (unordered) k-tuple of people from the group can be seated around a round table it the way that all pairs of neighbors are mutually friendly. *(Since this was the 67th year of CZE/SVK MO,)* show that if the group is 6-great, then it is 7-great as well.
Bonus (not included in the competition): Determine all positive integers $k \geq 3$ for which, if the group is k-great, then it is $(k+1)$-great as well.

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2 Let x, y, z be real numbers such that the numbers

$$\frac{1}{|x^2+2yz|}, \quad \frac{1}{|y^2+2zx|}, \quad \frac{1}{|z^2+2xy|}$$

are lengths of sides of a (non-degenerate) triangle. Determine all possible values of $xy+yz+zx$.

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3 In triangle ABC let be D an intersection of BC and the A-angle bisector. Denote E, F the circumcenters of ABD and ACD respectively. Assuming that the circumcenter of AEF lies on the line BC what is the possible size of the angle BAC ?

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4 Let a, b, c be integers which are lengths of sides of a triangle, $\gcd(a, b, c) = 1$ and all the values

$$\frac{a^2+b^2-c^2}{a+b-c}, \quad \frac{b^2+c^2-a^2}{b+c-a}, \quad \frac{c^2+a^2-b^2}{c+a-b}$$

are integers as well. Show that $(a+b-c)(b+c-a)(c+a-b)$ or $2(a+b-c)(b+c-a)(c+a-b)$ is a perfect square.

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5 Let $ABCD$ an isosceles trapezoid with the longer base AB. Denote I the incenter of ΔABC and J the excenter relative to the vertex C of ΔACD. Show that the lines IJ and AB are parallel.

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6 Determine the least positive integer n with the following property for every 3-coloring of numbers $1, 2, \ldots, n$ there are two (different) numbers a, b of the same color such that $|a-b|$ is a perfect square.
