Finals 2018

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–	**Day 1**
1	An acute triangle ABC in which $AB < AC$ is given. The bisector of $\angle BAC$ crosses BC at D. Point M is the midpoint of BC. Prove that the line though centers of circles escribed on triangles ABC and ADM is parallel to AD.
2	A subset S of size n of a plane consisting of points with both coordinates integer is given, where n is an odd number. The injective function $f\colon S \to S$ satisfies the following: for each pair of points $A, B \in S$, the distance between points $f(A)$ and $f(B)$ is not smaller than the distance between points A and B. Prove there exists a point X such that $f(X) = X$.
3	Find all real numbers c for which there exists a function $f\colon \mathbb{R} \to \mathbb{R}$ such that for each $x, y \in \mathbb{R}$ it's true that $$f(f(x) + f(y)) + cxy = f(x + y).$$

–	**Day 2**
4	Let n be a positive integer. Suppose there are exactly M squarefree integers k such that $\left\lfloor \frac{n}{k} \right\rfloor$ is odd in the set $\{1, 2, \ldots, n\}$. Prove M is odd. An integer is *squarefree* if it is not divisible by any square other than 1.
5	An acute triangle ABC in which $AB < AC$ is given. Points E and F are feet of its heights from B and C, respectively. The line tangent in point A to the circle escribed on ABC crosses BC at P. The line parallel to BC that goes through point A crosses EF at Q. Prove PQ is perpendicular to the median from A of triangle ABC.
6	A prime $p > 3$ is given. Let K be the number of such permutations $(a_1, a_2, \ldots, a_p)$ of $\{1, 2, \ldots, p\}$ such that $$a_1a_2 + a_2a_3 + \ldots + a_{p-1}a_p + a_pa_1$$ is divisible by p. Prove $K + p$ is divisible by p^2.