www.artofproblemsolving.com/community/c644917
by niyu, BarishNamazov, microsoft_office_word

– Day 1

1 Let ABC be a triangle with $CA = CB$ and $\angle ACB = 120^\circ$, and let M be the midpoint of AB. Let P be a variable point of the circumcircle of ABC, and let Q be the point on the segment CP such that $QP = 2QC$. It is given that the line through P and perpendicular to AB intersects the line MQ at a unique point N.
Prove that there exists a fixed circle such that N lies on this circle for all possible positions of P.

2 Consider the set
$$A = \left\{1 + \frac{1}{k} : k = 1, 2, 3, 4, \cdots \right\}.$$

-Prove that every integer $x \geq 2$ can be written as the product of one or more elements of A, which are not necessarily different.

-For every integer $x \geq 2$ let $f(x)$ denote the minimum integer such that x can be written as the product of $f(x)$ elements of A, which are not necessarily different.
Prove that there exist infinitely many pairs (x, y) of integers with $x \geq 2$, $y \geq 2$, and
$$f(xy) < f(x) + f(y).$$
(Pairs (x_1, y_1) and (x_2, y_2) are different if $x_1 \neq x_2$ or $y_1 \neq y_2$).

3 The n contestant of EGMO are named $C_1, C_2, \cdots C_n$. After the competition, they queue in front of the restaurant according to the following rules.

-The Jury chooses the initial order of the contestants in the queue.
-Every minute, the Jury chooses an integer i with $1 \leq i \leq n$.

-If contestant C_i has at least i other contestants in front of her, she pays one euro to the Jury and moves forward in the queue by exactly i positions.
-If contestant C_i has fewer than i other contestants in front of her, the restaurant opens and process ends.

-Prove that the process cannot continue indefinitely, regardless of the Jurys choices.
-Determine for every n the maximum number of euros that the Jury can collect by cunningly choosing the initial order and the sequence of moves.

– Day 2

4 A domino is a 1×2 or 2×1 tile.
Let $n \geq 3$ be an integer. Dominoes are placed on an $n \times n$ board in such a way that each domino covers exactly two cells of the board, and dominoes do not overlap. The value of a row or column is the number of dominoes that cover at least one cell of this row or column. The configuration is called balanced if there exists some $k \geq 1$ such that each row and each column has a value of k. Prove that a balanced configuration exists for every $n \geq 3$, and find the minimum number of dominoes needed in such a configuration.

---

5 Let Γ be the circumcircle of triangle ABC. A circle Ω is tangent to the line segment AB and is tangent to Γ at a point lying on the same side of the line AB as C. The angle bisector of $\angle BCA$ intersects Ω at two different points P and Q.
Prove that $\angle ABP = \angle QBC$.

6

-Prove that for every real number t such that $0 < t < \frac{1}{2}$ there exists a positive integer n with the following property: for every set S of n positive integers there exist two different elements x and y of S, and a non-negative integer m (i.e. $m \geq 0$), such that

$$|x - my| \leq ty.$$

-Determine whether for every real number t such that $0 < t < \frac{1}{2}$ there exists an infinite set S of positive integers such that

$$|x - my| > ty$$

for every pair of different elements x and y of S and every positive integer m (i.e. $m > 0$).
