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Day 1 April 18

1 Let a, b, c be positive real numbers such that $a+b+c=4\sqrt[3]{abc}$. Prove that

$$2(ab+bc+ca)+4\min(a^2,b^2,c^2)\geq a^2+b^2+c^2.$$

2 Find all functions $f:(0,\infty)\rightarrow(0,\infty)$ such that

$$f\left(x+\frac{1}{y}\right)+f\left(y+\frac{1}{z}\right)+f\left(z+\frac{1}{x}\right)=1$$

for all $x, y, z>0$ with $xyz=1$.

3 For a given integer $n\geq 2$, let $\{a_1,a_2,,a_m\}$ be the set of positive integers less than n that are relatively prime to n. Prove that if every prime that divides m also divides n, then $a_1^k+a_2^k+\cdots+a_m^k$ is divisible by m for every positive integer k.

Proposed by Ivan Borsenco

Day 2 April 19

4 Let p be a prime, and let $a_1,\ldots,a_p$ be integers. Show that there exists an integer k such that the numbers

$$a_1+k, a_2+2k,\ldots,a_p+pk$$

produce at least $\frac{1}{2}p$ distinct remainders upon division by p.

Proposed by Ankan Bhattacharya

5 In convex cyclic quadrilateral $ABCD$, we know that lines AC and BD intersect at E, lines AB and CD intersect at F, and lines BC and DA intersect at G. Suppose that the circumcircle of $\triangle ABE$ intersects line CB at B and P, and the circumcircle of $\triangle ADE$ intersects line CD at D and Q, where C, B, P, G and C, Q, D, F are collinear in that order. Prove that if lines FP and GQ intersect at M, then $\angle MAC=90^\circ$.

Proposed by Kada Williams

6 Let a_n be the number of permutations $(x_1, x_2, \dots, x_n)$ of the numbers $(1, 2, \dots, n)$ such that the n ratios $\frac{x_k}{k}$ for $1 \le k \le n$ are all distinct. Prove that a_n is odd for all $n \ge 1$.

Proposed by Richard Stong