National Math Olympiad (Second Round)

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1 Let P be the intersection of AC and BD in isosceles trapezoid $ABCD$ ($AB \parallel CD$, $BC = AD$) . The circumcircle of triangle ABP inersects BC for the second time at X. Point Y lies on AX such that $DY \parallel BC$. Prove that $Y\hat{D}A = 2.Y\hat{C}A$.

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2 Let n be odd natural number and $x_1, x_2, \cdots, x_n$ be pairwise distinct numbers. Prove that someone can divide the difference of these number into two sets with equal sum.
($X = \{\mid x_i - x_j \mid |i < j\}$)

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3 Let $a > k$ be natural numbers and $r_1 < r_2 < \ldots r_n, s_1 < s_2 < \cdots < s_n$ be sequences of natural numbers such that:

$(a^{r_1} + k)(a^{r_2} + k) \ldots (a^{r_n} + k) = (a^{s_1} + k)(a^{s_2} + k) \ldots (a^{s_n} + k)$

Prove that these sequences are equal.

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4 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that:

$$f(x + y)f(x^2 - xy + y^2) = x^3 + y^3$$

for all reals x, y.

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5 Lamps of the hall switch by only five keys. Every key is connected to one or more lamp(s). By switching every key, all connected lamps will be switched too. We know that no two keys have same set of connected lamps with each other. At first all of the lamps are off. Prove that someone can switch just three keys to turn at least two lamps on.

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6 Two circles ω_1, ω_2 intersect at P, Q. An arbitrary line passing through P intersects ω_1, ω_2 at A, B respectively. Another line parallel to AB intersects ω_1 at D, F and ω_2 at E, C such that E, F lie between C, D.Let $X \equiv AD \cap BE$ and $Y \equiv BC \cap AF$. Let R be the reflection of P about CD. Prove that:
a. R lies on XY.
b. PR is the bisector of $X\hat{P}Y$.
