Kazakhstan National Olympiad 2018

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by CinarArslan, qweDota

– Grade 11

– [b] Day 1

1 In an equilateral trapezoid, the point O is the midpoint of the base AD. A circle with a center at a point O and a radius BO is tangent to a straight line AB. Let the segment AC intersect this circle at point $K(K \neq C)$, and let M is a point such that $ABCM$ is a parallelogram. The circumscribed circle of a triangle CMD intersects the segment AC at a point $L(L \neq C)$. Prove that $AK = CL$.

2 The natural number $m \geq 2$ is given.Sequence of natural numbers $(b_0, b_1, \ldots, b_m)$ is called concave if $b_k + b_{k-2} \leq 2b_{k-1}$ for all $2 \leq k \leq m$. Prove that there exist not greater than 2^m concave sequences starting with $b_0 = 1$ or $b_0 = 2$

3 Is there exist a function $f : \mathbb{N} \to \mathbb{N}$ with for $\forall m, n \in \mathbb{N}$

$$f(mf(n)) = f(m)f(m+n) + n?$$

– [b] Day 2

4 Prove that for all reas $a, b, c, d \in (0, 1)$ we have

$$(ab - cd)(ac + bd)(ad - bc) + \min(a, b, c, d) < 1.$$

5 Given set $S = \{xy(x+y) \mid x, y \in \mathbb{N}\}$.Let a and n natural numbers such that $a + 2^k \in S$ for all $k = 1, 2, 3, ..., n$.Find the greatest value of n.

6 Inside of convex quadrilateral $ABCD$ found a point M such that $\angle AMB = \angle ADM + \angle BCM$ and $\angle AMD = \angle ABM + \angle DCM$.Prove that

$$AM \cdot CM + BM \cdot DM \geq \sqrt{AB \cdot BC \cdot CD \cdot DA}.$$