**Saint Petersburg Mathematical Olympiad 2017**

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| – | Grade 9 |
| 1 | Sashas computer can do the following two operations: If you load the card with number $a$, it will return that card back and also prints another card with number $a+1$, and if you consecutively load the cards with numbers $a$ and $b$, it will return them back and also prints cards with all the roots of the quadratic trinomial $x^2+ax+b$ (possibly one, two, or none cards.) Initially, Sasha had only one card with number $s$. Is it true that, for any $s>0$, Sasha can get a card with number $\sqrt{s}$? |
| 2 | Given a triangle $ABC$, theres a point $X$ on the side $AB$ such that $2BX = BA + BC$. Let $Y$ be the point symmetric to the incenter $I$ of triangle $ABC$, with respect to point $X$. Prove that $YI_B \perp AB$ where $I_B$ is the $B$-excenter of triangle $ABC$. |
| 3 | Petya, Vasya and Tolya play a game on a $100\times 100$ board. They take turns (starting from Petya, then Vasya, then Tolya, then Petya, etc.) paint the boundary cells of the board (i.e., having a common side with the boundary of the board.) It is forbidden to paint the cell that is adjacent to the already painted one. In addition, its also forbidden to paint the cell which is symmetrical to the painted one, with respect to the center of the board. The player who cant make the move loss. Can Vasya and Tolya, after agreeing, play so that Petya loses? |
| 4 | Each cell of a $3\times n$ table was filled by a number. In each of three rows, the number $1, 2, , n$ appear in some order. It is know that for each column, the sum of pairwise product of three numbers in it is a multiple of $n$. Find all possible value of $n$. |
| 5 | Given a scalene triangle $ABC$ with $\angle B = 130^\circ$. Let $H$ be the foot of altitude from $B$. $D$ and $E$ are points on the sides $AB$ and $BC$, respectively, such that $DH = EH$ and $ADEC$ is a cyclic quadrilateral. Find $\angle DHE$. |
| 6 | Given three real numbers $a, b, c \in [0, 1)$ such that $a^2 + b^2 + c^2 = 1$. Find the smallest possible value of $$\frac{a}{\sqrt{1-a^2}}+\frac{b}{\sqrt{1-b^2}}+\frac{c}{\sqrt{1-c^2}}.$$ |
| 7 | Divide the upper right quadrant of the plane into square cells with side length $1$. In this quadrant, $n^2$ cells are colored, show that therere at least $n^2+n$ cells (possibly including the colored ones) that at least one of its neighbors are colored. |

– Grade 10

**1** Its allowed to replace any of three coefficients of quadratic trinomial by its discriminant. Is it true that from any quadratic trinomial that does not have real roots, we can perform such operation several times to get a quadratic trinomial that have real roots?

**2** $(a_n)$ is sequence with positive integer. $a_1 > 10$ $a_n = a_{n-1} + GCD(n, a_{n-1})$, n¿1
For some i $a_i = 2i$.
Prove that these numbers are infinite in this sequence.

**3** Let $ABC$ be an acute triangle, with median $AM$, height $AH$ and internal angle bisector $AL$. Suppose that $B, H, L, M, C$ are collinear in that order, and $LH < LM$. Prove that $BC > 2AL$.

**4** The numbers from $1$ to $2000^2$ were written on a board. Vasya choose $2000$ of them whose sum of them equal to two thousandth of the sum of all numbers. Proof that his friend, Petya, will be able to color each of the remaining numbers by one of other $1999$ colors so that the sum of numbers with each of total $2000$ colors are the same.

**5** Let $x, y, z > 0$ and $\sqrt{xyz} = xy + yz + zx$. Prove that

$$x + y + z \leq \frac{1}{3}.$$

**6** In acute-angled triangle $ABC$, the height $AH$ and median $BM$ were drawn. Point $D$ lies on the circumcircle of triangle $BHM$ such that $AD \parallel BM$ and $B, D$ are on opposite sides of line $AC$. Prove that $BC = BD$.

**7** In a country, some pairs of cities are connected by one-way roads. It turns out that every city has at least two out-going and two in-coming roads assigned to it, and from every city one can travel to any other city by a sequence of roads. Prove that it is possible to delete a cyclic route so that it is still possible to travel from any city to any other city.

– Grade 11

**1** A1,A2,...,Am are subsets of X and we have —Ai—=mk (m,k natural numbers)
prove that we can separate X into k sets such that every set has at least one member of each Ai.

**2** A circle passing through vertices $A$ and $B$ of triangle $ABC$ intersects the sides $AC$ and $BC$ again at points $P$ and $Q$, respectively. Given that the median from vertex $C$ bisect the arc $PQ$ of the circle. Prove that $ABC$ is an isosceles triangle.

**3** Given real numbers $x, y, z, t \in (0, \pi/2]$ such that

$$\cos^2(x) + \cos^2(y) + \cos^2(z) + \cos^2(t) = 1.$$

What is the minimum possible value of

$$\cot(x) + \cot(y) + \cot(z) + \cot(t)?$$

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**4** A positive integer $n$ is called almost-square if $n$ can be represented as $n = ab$ where $a, b$ are positive integers that $a \le b \le 1.01a$. Prove that there exists infinitely many positive integers $m$ that therere no almost-square positive integer among $m, m+1, , m+198$.

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**5** Given a tetrahedron $PABC$, draw the height $PH$ from vertex $P$ to $ABC$. From point $H$, draw perpendiculars $HA, HB, HC$ to the lines $PA, PB, PC$. Suppose the planes $ABC$ and $ABC$ intersects at line $\ell$. Let $O$ be the circumcenter of triangle $ABC$. Prove that $OH \perp \ell$.

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**6** In the country some mathematicians know each other and any division of them into two sets contain 2 friends from different sets.It is known that if you put any set of four or more mathematicians at a round table so that any two neighbours know each other , then at the table there are two friends not sitting next to each other.We denote by $c_i$ the number of sets of $i$ pairwise familiar mathematicians(by saying "familiar" it means know each other).Prove that $c_1 - c_2 + c_3 - c_4 + ... = 1$

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**7** Given a convex polygon with vertices at lattice points on a plane containing origin $O$. Let $V_1$ be the set of vectors going from $O$ to the vertices of the polygon, and $V_2$ be the set of vectors going from $O$ to the lattice points that lie inside or on the boundary of the polygon (thus, $V_1$ is contained in $V_2$.) Two grasshoppers jump on the whole plane: each jump of the first grasshopper shift its position by a vector from the set $V_1$, and the second by the set $V_2$. Prove that there exists positive integer $c$ that the following statement is true: if both grasshoppers can jump from $O$ to some point $A$ and the second grasshopper needs $n$ jumps to do it, then the first grasshopper can use at most $n + c$ jumps to do so.

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