Moroccan Team Selection Test 2005

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–	Day 1
1	Prove that the equation $3y^2 = x^4 + x$ has no positive integer solutions.
2	Let A be a set of positive integers such that a) if $a \in A$, the all the positive divisors of a are also in A; b) if $a, b \in A$, with $1 < a < b$, then $1 + ab \in A$. Prove that if A has at least 3 elements, then A is the set of all positive integers.
3	The real numbers $a_1, a_2, ..., a_{100}$ satisfy the relationship : $a_1^2 + a_2^2 + \cdots + a_{100}^2 + (a_1 + a_2 + \cdots + a_{100})^2 = 101$ Prove that $\lvert a_k \rvert \leq 10$ for all $k \in \{1, 2, ..., 100\}$
4	Consider a cyclic quadrilateral $ABCD$, and let S be the intersection of AC and BD. Let E and F the orthogonal projections of S on AB and CD respectively. Prove that the perpendicular bisector of segment EF meets the segments AD and BC at their midpoints.
–	Day 2
1	Find all the functions $f : \mathbb{R} \to \mathbb{R}$ satisfying : $(x+y)(f(x) - f(y)) = (x-y)f(x+y)$ for all $x, y \in \mathbb{R}$
2	Let a, b, c be positive real numbers. Prove the inequality $$\frac{a^2}{b} + \frac{b^2}{c} + \frac{c^2}{a} \geq a + b + c + \frac{4(a-b)^2}{a+b+c}.$$ When does equality occur?
3	Find all primes p such that $p^2 - p + 1$ is a perfect cube.
4	A convex quadrilateral $ABCD$ has an incircle. In each corner a circle is inscribed that also externally touches the two circles inscribed in the adjacent corners. Show that at least two circles have the same size.

– Day 3

1 Find all the positive primes p for which there exist integers m, n satisfying : $p = m^2 + n^2$ and $m^3 + n^3 - 4$ is divisible by p.

2 Consider the set $A = \{1, 2, ..., 49\}$. We partitionate A into three subsets. Prove that there exist a set from these subsets containing three distincts elements a, b, c such that $a + b = c$

3 Let $a_1, a_2, \ldots$ be an infinite sequence of real numbers, for which there exists a real number c with $0 \leq a_i \leq c$ for all i, such that

$$|a_i - a_j| \geq \frac{1}{i+j} \quad \text{for all } i,\ j \text{ with } i \neq j.$$

Prove that $c \geq 1$.

4 Let $ABCD$ be a cyclic qudrilaterlal such that $AB.BC = 2.CD.DA$
Prove that $8.BD^2 \leq 9.AC^2$