Moroccan Team Selection Test 2011

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by tchebytchev, orl, Martin N., Amir Hossein

– Day 1

1 Prove that for any n natural, the number

$$\sum_{k=0}^{n}\binom{2n+1}{2k+1}2^{3k}$$

cannot be divided by 5.

2 For positive integers m and n, find the smalles possible value of $|2011^m - 45^n|$.

(Swiss Mathematical Olympiad, Final round, problem 3)

3 For a given triangle ABC, let X be a variable point on the line BC such that C lies between B and X and the incircles of the triangles ABX and ACX intersect at two distinct points P and Q. Prove that the line PQ passes through a point independent of X.

– Day 2

1 Find all pairs (m, n) of nonnegative integers for which

$$m^2 + 2\cdot 3^n = m\left(2^{n+1} - 1\right).$$

Proposed by Angelo Di Pasquale, Australia

2 Let $x_1, \ldots, x_{100}$ be nonnegative real numbers such that $x_i + x_{i+1} + x_{i+2} \leq 1$ for all $i = 1, \ldots, 100$ (we put $x_{101} = x_1, x_{102} = x_2$). Find the maximal possible value of the sum $S = \sum_{i=1}^{100} x_i x_{i+2}$.

Proposed by Sergei Berlov, Ilya Bogdanov, Russia

3 The vertices X, Y, Z of an equilateral triangle XYZ lie respectively on the sides BC, CA, AB of an acute-angled triangle ABC. Prove that the incenter of triangle ABC lies inside triangle XYZ.

Proposed by Nikolay Beluhov, Bulgaria