AoPS Online
Art of Problem Solving

Moroccan Team Selection Test 2010

www.artofproblemsolving.com/community/c65667

by tchebytchev, Valentin Vornicu, Rushil, orl, Amir Hossein, mattilgale

– Day 1

1 f is a function twice differentiable on $[0,1]$ and such that f'' is continuous. We suppose that : $f(1)-1=f(0)=f'(1)=f'(0)=0$.
Prove that there exists x_0 on $[0,1]$ such that $|f''(x_0)|\geq 4$

2 Let a, b, c be positive real numbers with $abc\leq a+b+c$. Show that

$$a^2+b^2+c^2\geq\sqrt{3}abc.$$

Cristinel Mortici, Romania

3 Any rational number admits a non-decimal representation unlimited decimal expansion. This development has the particularity of being periodic.
Examples: $\frac{1}{7}=0.142857142857$ has a period 6 while $\frac{1}{11}=0.0909090909$ 2 periodic.
What are the reciprocals of the prime integers with a period less than or equal to five?

4 Find all triangles whose side lengths are consecutive integers, and one of whose angles is twice another.

– Day 2

1 In a sports meeting a total of m medals were awarded over n days. On the first day one medal and $\frac{1}{7}$ of the remaining medals were awarded. On the second day two medals and $\frac{1}{7}$ of the remaining medals were awarded, and so on. On the last day, the remaining n medals were awarded. How many medals did the meeting last, and what was the total number of medals ?

2 Find the integer represented by $\left[\sum_{n=1}^{10^9}n^{-2/3}\right]$. Here $[x]$ denotes the greatest integer less than or equal to x.

3 Let G be a non-empty set of non-constant functions f such that $f(x)=ax+b$ (where a and b are two reals) and satisfying the following conditions:
1) if $f\in G$ and $g\in G$ then $gof\in G$,
2) if $f\in G$ then $f^{-1}\in G$,
3) for all $f\in G$ there exists $x_f\in\mathbb{R}$ such that $f(x_f)=x_f$.
Prove that there is a real k such that for all $f\in G$ we have $f(k)=k$

4 Let $ABCDE$ be a convex pentagon such that

$$\angle BAC = \angle CAD = \angle DAE \qquad \text{and} \qquad \angle ABC = \angle ACD = \angle ADE.$$

The diagonals BD and CE meet at P. Prove that the line AP bisects the side CD.

Proposed by Zuming Feng, USA