Azerbaijan Team Selection Test 2017

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by IstekOlympiadTeam, cjquines0, mathwizard888, GGPiku

Day 1 May 13rd

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1 Find all positive integers n for which all positive divisors of n can be put into the cells of a rectangular table under the following constraints:

-each cell contains a distinct divisor;
-the sums of all rows are equal; and
-the sums of all columns are equal.

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2 Let ABC be a triangle with $AB = AC \neq BC$ and let I be its incentre. The line BI meets AC at D, and the line through D perpendicular to AC meets AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.

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3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + yf(x^2)) = f(x) + xf(xy)$$

for all real numbers x and y.

Day 2 May 14th

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1 Let ABC be an acute angled triangle. Points E and F are chosen on the sides AC and AB, respectively, such that

$$BC^2 = BA \times BF + CE \times CA.$$

Prove that for all such E and F, circumcircle of the triangle AEF passes through a fixed point different from A.

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3 Consider fractions $\frac{a}{b}$ where a and b are positive integers.
(a) Prove that for every positive integer n, there exists such a fraction $\frac{a}{b}$ such that $\sqrt{n} \leq \frac{a}{b} \leq \sqrt{n+1}$ and $b \leq \sqrt{n} + 1$.
(b) Show that there are infinitely many positive integers n such that no such fraction $\frac{a}{b}$ satisfies $\sqrt{n} \leq \frac{a}{b} \leq \sqrt{n+1}$ and $b \leq \sqrt{n}$.

Day 1 May 15th

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1 Consider the sequence of rational numbers defined by $x_1 = \frac{4}{3}$, and $x_{n+1} = \frac{x_n^2}{x_n^2 - x_n + 1}$. Show that the nu,erator of the lowest term expression of each sum $x_1 + x_2 + ... + x_k$ is a perfect square.

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2 Let n, m, k and l be positive integers with $n \neq 1$ such that $n^k + mn^l + 1$ divides $n^{k+l} - 1$. Prove that

-$m = 1$ and $l = 2k$; or
-$l|k$ and $m = \frac{n^{k-l}-1}{n^l-1}$.

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3 Let n be a positive integer relatively prime to 6. We paint the vertices of a regular n-gon with three colours so that there is an odd number of vertices of each colour. Show that there exists an isosceles triangle whose three vertices are of different colours.
