German National Olympiad 2018, Final Round

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by Tintarn

–	Day 1
1	Find all real numbers x, y, z satisfying the following system of equations: $$xy + z = -30$$ $$yz + x = 30$$ $$zx + y = -18$$
2	We are given a tetrahedron with two edges of length a and the remaining four edges of length b where a and b are positive real numbers. What is the range of possible values for the ratio $v = a/b$?
3	Given a positive integer n, Susann fills a square of $n \times n$ boxes. In each box she inscribes an integer, taking care that each row and each column contains distinct numbers. After this an imp appears and destroys some of the boxes. Show that Susann can choose some of the remaining boxes and colour them red, satisfying the following two conditions: 1) There are no two red boxes in the same column or in the same row. 2) For each box which is neither destroyed nor coloured, there is a red box with a larger number in the same row or a red box with a smaller number in the same column. *Proposed by Christian Reiher*
–	Day 2
4	a) Let a, b and c be side lengths of a triangle with perimeter 4. Show that $$a^2 + b^2 + c^2 + abc < 8.$$ b) Is there a real number $d < 8$ such that for all triangles with perimeter 4 we have $$a^2 + b^2 + c^2 + abc < d$$ where a, b and c are the side lengths of the triangle?
5	We define a sequence of positive integers $a_1, a_2, a_3, \dots$ as follows: Let $a_1 = 1$ and iteratively, for $k = 2, 3, \dots$ let a_k be the largest prime factor of $1 + a_1 a_2 \cdots a_{k-1}$. Show that the number 11 is not an element of this sequence.

6 Let P be a point in the interior of a triangle ABC and let the rays $\overrightarrow{AP}, \overrightarrow{BP}$ and $\overrightarrow{CP}$ intersect the sides BC, CA and AB in A_1, B_1 and C_1, respectively. Let D be the foot of the perpendicular from A_1 to B_1C_1. Show that

$$\frac{CD}{BD} = \frac{B_1C}{BC_1} \cdot \frac{C_1A}{AB_1}.$$
