Olympic Revenge 2018

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1 Let $(F_n)_{n\geq 1}$ the Fibonacci sequence. Find all $n \in \mathbb{N}$ such that for every $k=0,1,...,F_n$

$$\binom{F_n}{k} \equiv (-1)^k \ (mod\ F_n+1)$$

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2 Let $\triangle ABC$ a scalene triangle with incenter I, circumcenter O and circumcircle Γ. The incircle of $\triangle ABC$ is tangent to BC, CA and AB at points D, E and F, respectively. The line AI meet EF and Γ at N and $M \neq A$, respectively. MD meet Γ at $L \neq M$ and IL meet EF at K. The circumference of diameter MN meet Γ at $P \neq M$. Prove that AK, PN and OI are concurrent.

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3 In a mathematical challenge, positive real numbers $a_1 \geq a_2 \geq ... \geq a_n$ and an initial sequence of positive real numbers $(b_1, b_2, ..., b_{n+1})$ are given to Secco. Let C a non-negative real number. In a sequence $(x_1, x_2, ..., x_{n+1})$, consider the following operation:
Subtract 1 of some x_j, $j \in \{1,2,...,n+1\}$, add C to x_{n+1} and replace $(x_1, x_2, ..., x_{j-1})$ for $(x_1+a_{\sigma(1)}, x_2+a_{\sigma(2)}, ..., x_{j-1}+a_{\sigma(j-1)})$, where σ is a permutation of $(1,2,...,j-1)$.
Secco's goal is to make all terms of sequence (b_k) negative after a finite number of operations. Find all values of C, depending of $a_1, a_2, ..., a_n, b_1, b_2, ..., b_{n+1}$, for which Secco can attain his goal.

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4 Let $\triangle ABC$ an acute triangle of incenter I and incircle ω. ω is tangent to BC, CA and AB at points T_A, T_B and T_C, respectively. Let l_A the line through A and parallel to BC and define l_B and l_C analogously. Let L_A the second intersection point of AI with the circumcircle of $\triangle ABC$ and define L_B and L_C analogously. Let $P_A = T_BT_C \cap l_A$ and define P_B and P_C analogously. Let $S_A = P_BT_B \cap P_CT_C$ and define S_B and S_C analogously. Prove that S_AL_A, S_BL_B, S_CL_C are concurrent.

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5 Let p a positive prime number and $\mathbb{F}_p$ the set of integers $mod\ p$. For $x \in \mathbb{F}_p$, define $|x|$ as the cyclic distance of x to 0, that is, if we represent x as an integer between 0 and $p-1$, $|x|=x$ if $x<\frac{p}{2}$, and $|x|=p-x$ if $x>\frac{p}{2}$. Let $f: \mathbb{F}_p \to \mathbb{F}_p$ a function such that for every $x, y \in \mathbb{F}_p$

$$|f(x+y)-f(x)-f(y)| < 100$$

Prove that exist $m \in \mathbb{F}_p$ such that for every $x \in \mathbb{F}_p$

$$|f(x)-mx| < 1000$$
