20th Centroamerican and Caribbean Math Olympiad, Havana 2018

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by jestrada

1 There are 2018 cards numbered from 1 to 2018. The numbers of the cards are visible at all times. Tito and Pepe play a game. Starting with Tito, they take turns picking cards until they're finished. Then each player sums the numbers on his cards and whoever has an even sum wins. Determine which player has a winning strategy and describe it.

P.S. Proposed by yours truly :-D

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2 Let ΔABC be a triangle inscribed in the circumference ω of center O. Let T be the symmetric of C respect to O and T' be the reflection of T respect to line AB. Line BT' intersects ω again at R. The perpendicular to CT through O intersects line AC at L. Let N be the intersection of lines TR and AC. Prove that $\overline{CN} = 2\overline{AL}$.

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3 Let x, y be real numbers such that $x - y, x^2 - y^2, x^3 - y^3$ are all prime numbers. Prove that $x - y = 3$.

EDIT: Problem submitted by Leonel Castillo, Panama.

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4 Determine all triples (p, q, r) of positive integers, where p, q are also primes, such that $\frac{r^2-5q^2}{p^2-1} = 2$.

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5 Let n be a positive integer, $1 < n < 2018$. For each $i = 1, 2, \ldots, n$ we define the polynomial $S_i(x) = x^2 - 2018x + l_i$, where $l_1, l_2, \ldots, l_n$ are distinct positive integers. If the polynomial $S_1(x) + S_2(x) + \cdots + S_n(x)$ has at least an integer root, prove that at least one of the l_i is greater or equal than 2018.

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6 A dance with 2018 couples takes place in Havana. For the dance, 2018 distinct points labeled $0, 1, \ldots, 2017$ are marked in a circumference and each couple is placed on a different point. For $i \geq 1$, let $s_i = i \pmod{2018}$ and $r_i = 2i \pmod{2018}$. The dance begins at minute 0. On the i-th minute, the couple at point s_i (if there's any) moves to point r_i, the couple on point r_i (if there's any) drops out, and the dance continues with the remaining couples. The dance ends after 2018^2 minutes. Determine how many couples remain at the end.

Note: If $r_i = s_i$, the couple on s_i stays there and does not drop out.
