ELMO Shortlist 2018

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by a1267ab, whatshisbucket

– Algebra

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1 Let $f:\mathbb{R}\to\mathbb{R}$ be a bijective function. Does there always exist an infinite number of functions $g:\mathbb{R}\to\mathbb{R}$ such that $f(g(x))=g(f(x))$ for all $x\in\mathbb{R}$?

Proposed by Daniel Liu

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2 Let $a_1,a_2,\dots,a_m$ be a finite sequence of positive integers. Prove that there exist nonnegative integers $b,c,$ and N such that

$$\left\lfloor \sum_{i=1}^m \sqrt{n+a_i} \right\rfloor = \left\lfloor \sqrt{bn+c} \right\rfloor$$

holds for all integers $n>N.$

Proposed by Carl Schildkraut

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3 Let a,b,c,x,y,z be positive reals such that $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$. Prove that

$$a^x+b^y+c^z\ge \frac{4abcxyz}{(x+y+z-3)^2}.$$

Proposed by Daniel Liu

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4 Elmo calls a monic polynomial with real coefficients *tasty* if all of its coefficients are in the range $[-1,1]$. A monic polynomial P with real coefficients and complex roots $\chi_1,\cdots,\chi_m$ (counted with multiplicity) is given to Elmo, and he discovers that there does not exist a monic polynomial Q with real coefficients such that PQ is tasty. Find all possible values of $\max\left(|\chi_1|,\cdots,|\chi_m|\right)$.

Proposed by Carl Schildkraut

– Combinatorics

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1 Let n be a positive integer. There are $2018n+1$ cities in the Kingdom of Sellke Arabia. King Mark wants to build two-way roads that connect certain pairs of cities such that for each city C and integer $1\le i\le 2018,$ there are exactly n cities that are a distance i away from $C.$ (The *distance* between two cities is the least number of roads on any path between the two cities.)

For which n is it possible for Mark to achieve this?

Proposed by Michael Ren

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2 We say that a positive integer n is m-*expressible* if it is possible to get n from some m digits and the six operations $+, -, \times, \div$, exponentiation ^, and concatenation $\oplus$. For example, 5625 is 3-expressible (in two ways): both $5 \oplus (5$^$4)$ and $(7 \oplus 5)$^2 yield 5625.

Does there exist a positive integer N such that all positive integers with N digits are $(N-1)$-expressible?

Proposed by Krit Boonsiriseth

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3 A *windmill* is a closed line segment of unit length with a distinguished endpoint, the *pivot*. Let S be a finite set of n points such that the distance between any two points of S is greater than c. A configuration of n windmills is *admissible* if no two windmills intersect and each point of S is used exactly once as a pivot.

An admissible configuration of windmills is initially given to Geoff in the plane. In one operation Geoff can rotate any windmill around its pivot, either clockwise or counterclockwise and by any amount, as long as no two windmills intersect during the process. Show that Geoff can reach any other admissible configuration in finitely many operations, where

(i) $c = \sqrt{3}$,

(ii) $c = \sqrt{2}$.

Proposed by Michael Ren

– Geometry

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1 Let ABC be an acute triangle with orthocenter H, and let P be a point on the nine-point circle of ABC. Lines BH, CH meet the opposite sides AC, AB at E, F, respectively. Suppose that the circumcircles $(EHP), (FHP)$ intersect lines CH, BH a second time at Q, R, respectively. Show that as P varies along the nine-point circle of ABC, the line QR passes through a fixed point.

Proposed by Brandon Wang

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2 Let ABC be a scalene triangle with orthocenter H and circumcenter O. Let P be the midpoint of $\overline{AH}$ and let T be on line BC with $\angle TAO = 90^\circ$. Let X be the foot of the altitude from O onto line PT. Prove that the midpoint of $\overline{PX}$ lies on the nine-point circle* of $\triangle ABC$.

*The nine-point circle of $\triangle ABC$ is the unique circle passing through the following nine points: the midpoint of the sides, the feet of the altitudes, and the midpoints of $\overline{AH}$, $\overline{BH}$, and $\overline{CH}$.

Proposed by Zack Chroman

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3 Let A be a point in the plane, and ℓ a line not passing through A. Evan does not have a straightedge, but instead has a special compass which has the ability to draw a circle through three distinct noncollinear points. (The center of the circle is *not* marked in this process.) Additionally, Evan can mark the intersections between two objects drawn, and can mark an arbitrary point on a given object or on the plane.

(i) Can Evan construct* the reflection of A over ℓ?

(ii) Can Evan construct the foot of the altitude from A to ℓ?

*To construct a point, Evan must have an algorithm which marks the point in finitely many steps.

Proposed by Zack Chroman

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4 Let $ABCDEF$ be a hexagon inscribed in a circle Ω such that triangles ACE and BDF have the same orthocenter. Suppose that segments BD and DF intersect CE at X and Y, respectively. Show that there is a point common to Ω, the circumcircle of DXY, and the line through A perpendicular to CE.

Proposed by Michael Ren and Vincent Huang

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5 Let scalene triangle ABC have altitudes AD, BE, CF and circumcenter O. The circumcircles of $\triangle ABC$ and $\triangle ADO$ meet at $P \neq A$. The circumcircle of $\triangle ABC$ meets lines PE at $X \neq P$ and PF at $Y \neq P$. Prove that $XY \parallel BC$.

Proposed by Daniel Hu

– Number Theory

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1 Determine all nonempty finite sets of positive integers $\{a_1, \dots, a_n\}$ such that $a_1 \cdots a_n$ divides $(x+a_1)\cdots(x+a_n)$ for every positive integer x.

Proposed by Ankan Bhattacharya

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2 Call a number n *good* if it can be expressed as $2^x + y^2$ for where x and y are nonnegative integers.
(a) Prove that there exist infinitely many sets of 4 consecutive good numbers.
(b) Find all sets of 5 consecutive good numbers.

Proposed by Michael Ma

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3 Consider infinite sequences $a_1, a_2, \dots$ of positive integers satisfying $a_1 = 1$ and

$$a_n \mid a_k + a_{k+1} + \cdots + a_{k+n-1}$$

for all positive integers k and n. For a given positive integer m, find the maximum possible value of a_{2m}.

Proposed by Krit Boonsiriseth

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4 Say a positive integer $n > 1$ is d-coverable if for each non-empty subset $S \subseteq \{0, 1, \ldots, n-1\}$, there exists a polynomial P with integer coefficients and degree at most d such that S is exactly the set of residues modulo n that P attains as it ranges over the integers. For each n, find the smallest d such that n is d-coverable, or prove no such d exists.

Proposed by Carl Schildkraut
